0.1 A Note on the derivation of the model of "Advertising, Intangible Investment, and Unpriced Entertainment,"
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free entertainment goods with advertising
M is entertainment good
max
\[ U(z, M) = z + (b - a_u) M - \frac{1}{2} M^2 \]
b, a_u > 0, z is numeraire good,
s.t. \( z + pM = y = 1 - M (c_M + \alpha') + pM \rightarrow z = 1 - M (c_M + \alpha') \)
p is price of entertainment good
\[ \Lambda = z + (b - a_u) M - \frac{1}{2} M^2 - \lambda (z + pM - y) \]
\[ \frac{d\Lambda}{dM} = b - a_u - M - \lambda p = 0 \rightarrow M = b - a_u - p \]
Firm max:
firm set price \( p_M \) for good and gets \( \alpha \) per unit sold from advertiser;
distribution cost per unit is \( \alpha' + c_M \)
Max \( \Pi = (p + \alpha - \alpha' - c_M) M \)
s.t. \( M = b - a_u - p \)
\[ \Pi = (p + \alpha - \alpha' - c_M) (b - a_u - p) \]
\[ \frac{d\Pi}{dp} = b - a_u - p - (p + \alpha - \alpha' - c_M) = b - a_u + c_M - (\alpha - \alpha') - 2p = 0 \rightarrow \]
\[ p = \frac{1}{2} (b - a_u + c_M - (\alpha - \alpha')) \]
\[ \frac{d^2\Pi}{dp^2} = -2 \]
Thus if \( \alpha - \alpha' > b - a_u + c_M \) then \( p \leq 0 \).
\[ p=0 \rightarrow M = b - a_u \]
\[ \Pi = (\alpha - \alpha' - c_M) (b - a_u) > (b - a_u)^2 \]
\[ U = z + (b - a_u) M - \frac{1}{2} M^2 = 1 - (b - a_u) (c_M + \alpha') + \frac{1}{2} (b - a_u)^2 = 1 + \frac{1}{2} (b - a_u) (b - a_u - c_M - \alpha') \]
\[ \text{note: } (c_M + \alpha') < \alpha - (b - a_u) \rightarrow \]
\[ U=1-(b-a_u)(c_M+\alpha')+(\frac{1}{2}(b-a_u)^2>1-(b-a_u)[\alpha-(b-a_u)]+\frac{1}{2}(b-a_u)^2= \]
\[ 1 - \alpha (b - a_u) + \frac{1}{2} (b - a_u)^2 \]
Cost to advertisers: \( \alpha M = \alpha (b - a_u) \)

transmission cost: \( c_M M = c_M (b - a_u) \)
direct advertising cost \( \alpha'M = \alpha' (b - a_u) \)
value of entertainment \( (\alpha - \alpha' - c_m) (b - a_u) \)
entertainment subsidy \( (\alpha - \alpha' - c_m) (b - a_u) \left( \frac{\alpha}{\alpha - c_M} \right) \)
total advt cost \( \alpha' (b - a_u) \left( \frac{\alpha}{\alpha - c_M} \right) \)
direct utility gain: \( (b - a_u) M - \frac{1}{2} M^2 = \frac{1}{2} (b - a_u)^2 \)

If advertising is absent, then:
\[ p = \frac{1}{2} (b + c_M) \]
\[ M = \frac{b - c_M}{2} \]
\[ \Pi = (p - c_M) (M) = \frac{1}{4} (b - c_M)^2 \]
\[ U(z, M) = z + bM - \frac{1}{2} M^2 \]
\[ z = 1 - Mc_M \]

Direct utility gain:
\[
\frac{bM - \frac{1}{2} M^2}{(b-c_M)(3b+c_M)} = \frac{b (\frac{b-c_M}{2}) - \frac{1}{2} (\frac{b-c_M}{2})^2}{8} = \frac{(b-c_M)(4b-(b-c_M))}{8} \]

Total utility gain:
\[
z + bM - \frac{1}{2} M^2 = 1 - Mc_M + bM - \frac{1}{2} M^2 = 1 + \frac{1}{2} (b - c_M)^2 - \frac{1}{2} (\frac{b-c_M}{2})^2 \]
\[
U = 1 + (b - c_M) \left( \frac{b-c_M}{2} - \frac{1}{2} (\frac{b-c_M}{2})^2 \right) = 1 + \frac{3}{8} (b - c_M)^2 \]

cost to consumers:
\[
pM = \frac{1}{4} (b^2 - c_M^2) \]
transmission cost:
\[
c_M M = c_M (b - c_M) \]
payment to entertainer:
\[
\frac{1}{4} (b - c_M)^2 \]