Nonparametric methods for the consumer characteristics model

(Working version)

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Abstract

We extend the nonparametric methods developed by Samuelson (1948), Houthakker (1950), Afriat (1967), Diewert (1973) and Varian (1982, 1983, 1984) to consumer characteristics models. We derive the minimal necessary and sufficient empirical conditions under which data on the market behaviour of heterogeneous, price-taking, individual consumers are nonparametrically consistent with the consumer characteristics model (Gorman (1956), Lancaster (1966), Muellbauer (1974)). Where these conditions hold, we show how information may be recovered on individual consumer's marginal valuations of product attributes. Where the conditions fail we highlight the role which the introduction of unobserved product attributes can play in rationalising the data. We apply these ideas to consumer panel data on the Danish milk market.

Key Words: Product characteristics, revealed preference.

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1 Introduction

The idea that consumers have preferences over the characteristics of market goods is central to much applied microeconomic work on price indices, quality change and product market innovation. The contribution of this paper is to make this idea nonparametrically testable, and to develop methods which will, if the data are indeed theory-consistent, reveal features of consumers' preferences over product attributes without the need for any assumptions regarding functional form or the extent or form of preference heterogeneity.

We extend the methods developed by in the literature on nonparametric testing[1] to consumer characteristics models. We derive the minimal necessary and sufficient empirical conditions under which data on the market behaviour of heterogeneous, price-taking, individual consumers are nonparametrically consistent with the consumer characteristics model (Gorman (1956), Lancaster (1966), Muellbauer (1974)). Where these conditions hold, we show how information may be recovered on individual consumer's marginal valuations of product attributes. Where the conditions fail we highlight the role which the introduction of unobserved product attributes can play in rationalising the data. We apply these ideas to consumer panel data on the Danish milk market.

2 Necessary and sufficient conditions for characteristics models

The pure consumer characteristics model supposes that preferences have the following structure

$$\begin{aligned} u(\mathbf{q}) &= v(\mathbf{z}) \\ \mathbf{z} &= \mathbf{z}(\mathbf{q}) = \left[z^1(\mathbf{q}), ..., z^J(\mathbf{q})\right]' \end{aligned}$$

where $\mathbf{q}$ is a $(K \times 1)$ vector of quantities of market goods, $\mathbf{z}$ is a $(J \times 1)$ vector of the levels of various product characteristics produced by those market goods, and $u(\mathbf{q})$ and $v(\mathbf{z})$ are utility functions defined over market goods and characteristics respectively. The consumer choice model is

$$\max_{\mathbf{q}} v(\mathbf{z}) \text{ subject to } \mathbf{z} = \mathbf{z}(\mathbf{q}) \text{ and } \mathbf{p}_t'\mathbf{q} \leq x_t$$

We ask, given a dataset $\{\mathbf{p}_t, \mathbf{q}_t, \mathbf{z}_t\}_{t=1,...,T}$, when are these data rationalisable by the characteristics model? To make clear exactly what we mean by *"rationalises"* we have the following definitions.

Definition 1. A utility function $u(\mathbf{q})$ *q-rationalises* the data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,...,T}$ if $u(\mathbf{q}_t) \geq u(\mathbf{q})$ for all $\mathbf{q}$ such that $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\mathbf{q}$.

Definition 2. A utility function $v(\mathbf{z})$ *z-rationalises* the data $\{\mathbf{p}_t, \mathbf{q}_t, \mathbf{z}_t\}_{t=1,...,T}$ if $v(\mathbf{z}_t) \geq v(\mathbf{z})$ for all $\mathbf{z}$ such that $\mathbf{z} = \mathbf{z}(\mathbf{q})$ and $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\mathbf{q}$.

[1] Samuelson (1948), Houthakker (1950), Hanoch and Rothschild (1972), Afriat (1967), Diewert (1973), Diewert and Parkan (1978) and Varian (1982, 1983, 1984).

The first definition is simply the statement of the principle of revealed preference, the second is the equivalent principle in the characteristics context where there are additional constraints in the form of the characteristics' production functions.

2.1 The linear characteristics model

The most typical version of the consumer characteristics model is one in which $\mathbf{z}$ is a linear function $\mathbf{z} = \mathbf{A}'\mathbf{q}$ where $\mathbf{A}$ is $(K \times J)$ matrix of constants with full *column* rank: rank$(\mathbf{A}) = J$.

Theorem 1: *The following statements are equivalent.*
(U) *there exists a utility function* $v(\mathbf{z})$ *which is continuous and concave in characteristics and where* $\mathbf{z} = \mathbf{A}'\mathbf{q}$ *where the matrix* $\mathbf{A}$ *is known, which z-rationalises the data* $\{\mathbf{p}_t, \mathbf{q}_t, \mathbf{z}_t\}_{t=1,...,T}$.
(A) *there exist numbers* $\{V_t, \lambda_t > 0, \boldsymbol{\pi}_t\}_{t=1,...,T}$ *such that*

$$V_s \leq V_t + \lambda_t \boldsymbol{\pi}_t' (\mathbf{z}_s - \mathbf{z}_t) \qquad \text{for all } s, t = 1, ..., T \tag{A.1}$$

$$\mathbf{p}_t' \geq \boldsymbol{\pi}_t' \mathbf{A}' \text{ with equality when } q_t^k > 0 \text{ for all } t = 1, ..., T \tag{A.2}$$

(G) *the data* $\{\boldsymbol{\pi}_t, \mathbf{z}_t\}_{t=1,...,T}$ *pass GARP for some choice of* $\boldsymbol{\pi}_t$ *such that* $\mathbf{p}_t' \geq \boldsymbol{\pi}_t' \mathbf{A}$.

Since the linear characteristics model is a special case of the standard preference-for-market-goods model, we have the following corollary.

Corollary 1: *If there exists a utility function* $v(\mathbf{z})$ *with* $\mathbf{z} = \mathbf{A}'\mathbf{q}$ *which z-rationalises the data then a utility function* $u(\mathbf{q}) = v(\mathbf{A}'\mathbf{q})$ *also exists, is concave, and q-rationalises the data.*

2.2 The non-linear characteristics model

Here we consider the non-linear characteristics model in which the characteristics levels are concave functions of market goods ($\mathbf{z} = \mathbf{z}(\mathbf{q})$), and utility is non-decreasing in characteristics. In what follows we suppose that the Jacobian $\boldsymbol{\nabla}\mathbf{z}(\mathbf{q}_t)$ is known.

Theorem 2: *The following statements are equivalent.*
(U) *there exists a utility function* $v(\mathbf{z})$ *which is non-decreasing, continuous and concave in characteristics and where* $\mathbf{z} = \mathbf{z}(\mathbf{q})$ *is a concave function and where the Jacobian matrix* $\boldsymbol{\nabla}\mathbf{z}(\mathbf{q}_t)$ *is known, which z-rationalises the data* $\{\mathbf{p}_t, \mathbf{q}_t, \mathbf{z}_t\}_{t=1,...,T}$.
(A) *there exist numbers* $\{V_t, \lambda_t > 0, \boldsymbol{\pi}_t\}_{t=1,...,T}$ *such that*

$$V_s \leq V_t + \lambda_t \boldsymbol{\pi}_t' (\mathbf{z}_s - \mathbf{z}_t) \qquad \text{for all } s, t = 1, ..., T \tag{A.1}$$

$$\boldsymbol{\Delta}\mathbf{z}_t' \boldsymbol{\pi}_t \leq \Delta \mathbf{q}_t' \mathbf{p}_t \text{ for all } s, t = 1, ..., T \tag{A.2}$$

where $\boldsymbol{\Delta}\mathbf{x}_t$ *denotes* $[(\mathbf{x}_1 - \mathbf{x}_t), ..., (\mathbf{x}_T - \mathbf{x}_t)]$.

(G) *the data* $\{\boldsymbol{\pi}_t, \mathbf{z}_t\}_{t=1,\dots,T}$ *pass GARP for some choice of* $\boldsymbol{\pi}_t$ *such that* $\mathbf{p}_t' \geq \boldsymbol{\pi}_t' \boldsymbol{\nabla} \mathbf{z}(\mathbf{q}_t)$.

As before we have a corollary which relates the non-linear characteristics model to the more general preferences-for-market-goods model.

Corollary 2: *If there exists a utility function* $v(\mathbf{z})$ *with* $\mathbf{z} = \mathbf{z}(\mathbf{q})$ *which z-rationalises the data then a utility function* $u(\mathbf{q}) = v(\mathbf{z}(\mathbf{q}))$ *also exists, is concave, and q-rationalises the data.*

2.3 Unobserved characteristics

We first consider a case which could be interpreted as one in which only a sub-set of product characteristics are observed. Suppose that the data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\dots,T}$ passes GARP, but that the conditions for z-rationalisation are not met when we consider the observed set of characteristics $\{\mathbf{z}_t\}_{t=1,\dots,T}$. There exists no characteristics model consistent with these characteristics which can rationalise the data, but, it seems that we can always invent some new characteristics such that the augmented "data" is rationalisable. We will concentrate on results for the linear characteristics model and make use of the following Lemmas.

Lemma 1. *The following statements are equivalent:*
1) *The vector equation* $\mathbf{p}_t = \mathbf{A}\boldsymbol{\pi}_t$ *has a unique solution for* $\boldsymbol{\pi}_t$ *given by* $\boldsymbol{\pi}_t = \mathbf{A}^{+}\mathbf{p}_t$ *where* $\mathbf{A}^{+}$ *is the Moore-Penrose inverse of* $\mathbf{A}$.
2) *rank*$\{\mathbf{A} : \mathbf{p}_t\} =$ *rank*$\{\mathbf{A}\} = J$.

Lemma 2. *GARP for the data* $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\dots,T}$ *and the condition rank*$\{\mathbf{A} : \mathbf{p}_t\} =$ *rank*$\{\mathbf{A}\}$ *for all t, together imply the existence of a utility function* $v(\mathbf{z})$ *which z-rationalises the data.*

This is an alternative condition for z-rationalisation of the linear model which applies when the consumer either consumes all goods in all periods or is on the point of doing so (in which case the inequality in condition $(A.2)$ in Theorem 1 becomes an equality and the rank condition in Lemma 1 implies the existence of $\boldsymbol{\pi}_t$ which satisfy it). We now consider the role of unobserved characteristics and have the following result.

Theorem 3: *If the data* $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\dots,T}$ *pass GARP then we can always define, at most,* $\min\{K, T\}$ *pseudo-characteristics in addition to the observed ones, such that there exists a utility function which z-rationalises the augmented set of characteristics.*

The key to this result is to notice that for any $\mathbf{A}$ matrix and $(K \times T)$ matrix $\mathbf{P} = [\mathbf{p}_1, ..., \mathbf{p}_T]$ matrix we can choose a $(J \times T)$ matrix $\boldsymbol{\Pi} = [\boldsymbol{\pi}_1, ..., \boldsymbol{\pi}_t]$ such that that

$$\mathbf{P} = \mathbf{A}\boldsymbol{\Pi} + \mathbf{E}$$

and $rank\{\mathbf{A} : \mathbf{E} : \mathbf{P}\} = rank\{\mathbf{A} : \mathbf{E}\}$. As long as $\boldsymbol{\Pi}$ is such that this rank condition holds it may be chosen arbitrarily – a straightforward way in which to guarantee the condition is to set $\boldsymbol{\Pi} = \mathbf{A}^{+}\mathbf{P}$ and then $\mathbf{E} \perp \mathbf{A}$ but simple random number generation for $\mathbf{\Pi}$ is also very likely to work. We have two equivalent formulations:

$$\mathbf{P} = [\mathbf{A} : \mathbf{E}] \begin{bmatrix} \mathbf{\Pi} \\ \mathbf{I}_T \end{bmatrix} \qquad \text{or} \qquad \mathbf{P} = [\mathbf{A} : \mathbf{I}_K] \begin{bmatrix} \mathbf{\Pi} \\ \mathbf{E} \end{bmatrix}$$

both of which may be written

$$\mathbf{P} = \overline{\mathbf{A}}\ \overline{\mathbf{\Pi}}$$

where $\overline{\mathbf{A}}$ is the augmented $\mathbf{A}$ matrix in either case and $\overline{\mathbf{\Pi}}$ the augmented $\mathbf{\Pi}$ matrix. In both cases we have

$$rank\left\{\overline{\mathbf{A}} : \mathbf{P}\right\} = rank\left\{\overline{\mathbf{A}}\right\}$$

The two equations are obviously equivalent but they bear two different interpretations. The first introduces T new characteristics for each good which are specific to the consumer and which have a marginal valuations of one everywhere. The second introduces K market-good-specific dummy characteristics with corresponding marginal valuations given by $\mathbf{E}$. Clearly the two interpretations are informationally equivalent and either way $\mathbf{E}$ is not identified by anything other than the initial method of choosing $\mathbf{\Pi}$. We can therefore add the lesser of K or T new characteristics and, since we have the necessary rank condition, Lemma 2 applies and we can z-rationalise these "data" on the basis that, for any goods which the consumer does not presently buy, they are just indifferent between purchase and non-purchase.

We now consider the case in which no characteristics are observed at all. In the linear case this is equivalent to having an unknown transformation matrix $\mathbf{A}$. In such a case we may wish to test whether there is *any* linear characteristics model that could be z-rationalisable with some given data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\ldots,T}$. We have the following theorem:

Theorem 4: *There exist characteristics $\mathbf{z}_t = \mathbf{A}'\mathbf{q}_t$ and a non-trivial characteristics model, in which all goods are either bought or on the verge of being bought, which* z*-rationalises the data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\ldots,T}$ if and only if the data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\ldots,T}$ pass GARP and $rank\{\mathbf{P}\} < K$.*

This means that if the rank of the price data is less than the number of goods, then we can construct a technology matrix and corresponding pseudo-characteristics such that the "data" $\{\mathbf{p}_t, \mathbf{q}_t, \mathbf{z}_t\}_{t=1,\ldots,T}$ are rationalisable. By "non-trivial" we mean a characteristics model with fewer characteristics than goods. It is clear that $rank\{\mathbf{P}\}$ can never be more than K and so if we were to allow $J = K$ we could trivially set $\mathbf{A}$ to be the identity matrix and then Theorem 4 collapses to testing whether the data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\ldots,T}$ pass GARP (or one of the equivalent standard Afriat-Diewert-Varian conditions).

3 Practicalities

The conditions given in, for example, Theorem 1 for a linear characteristics model are difficult to implement. For condition (G) we need to find the shadow prices to implement the GARP test and there is no known algorithm to do this in a finite number of steps[2]. The two conditions in (A) look more promising since they appear to have the form of the constraints in a linear programming problem. The problem is that there is a strict inequality ($\lambda_t > 0$) and also that they contains products of unknowns (in this case the λ_t's and the $\boldsymbol{\pi}_t$'s). It turns out, however, that we can give an alternative set of conditions that are equivalent to (A) in Theorem 1 that do take a standard form. We have:

Theorem 5. *The following statement is equivalent to condition (A) in Theorem 1. There exist numbers and vectors* $\left\{\tilde{V}_t, \tilde{\lambda}_t \geq 1, \tilde{\boldsymbol{\pi}}_t\right\}_{t=1,\dots,T}$ *such that*

$$\tilde{V}_s \leq \tilde{V}_t + \tilde{\boldsymbol{\pi}}_t' (\mathbf{z}_s - \mathbf{z}_t) \quad \text{for all } s,t = 1,...,T \tag{A.1*}$$

$$\tilde{\lambda}_t \mathbf{p}_t \geq \mathbf{A}\tilde{\boldsymbol{\pi}}_t \quad \text{with equality when } q_t^k > 0 \text{ for all } t = 1,...,T \tag{A.2*}$$

These conditions are now in standard linear programming form in the sense that the unknowns enter the constraints linearly, given prices, levels of characteristics and the matrix $\mathbf{A}$. Thus they can potentially be checked using any algorithm which is designed to find a feasible solution for a linear programming problem in a finite number of steps. Condition (A) in Theorem 2 can similarly be recast as a linear program. Nevertheless, it should be pointed out that in large problems (many goods, characteristics and observations), though this approach will yield a result in finite time, that time might still be rather long. It should also be noted that whilst this approach will recover a feasible set of marginal valuations[3] these are by no means necessarily unique. Under some circumstances, however, the practicalities involved in testing the conditions in, for example, Theorem 1 are more straightforward, faster and will recover unique values for the consumer's marginal valuation of characteristics. The conditions in Theorem 1 allows for non-purchase of market goods. For those market goods which are bought, (A.2) holds with equality. Denote the sub-vector of prices in period t for goods where $q_t^k > 0$ by $\widehat{\mathbf{p}}_t$ and the corresponding reduced $\mathbf{A}$ matrix by $\widehat{\mathbf{A}}$. Then $\widehat{\mathbf{p}}_t = \widehat{\mathbf{A}}\boldsymbol{\pi}_t$ and we have the following result.

Theorem 6. *If* $rank\left\{\widehat{\mathbf{A}} : \widehat{\mathbf{p}}_t\right\} = rank\left\{\widehat{\mathbf{A}}\right\}$ and $\text{rows}(\widehat{\mathbf{p}}_t) \geq J$ *then* $\boldsymbol{\pi}_t = \widehat{\mathbf{A}}_t^+ \widehat{\mathbf{p}}$ *where* $\widehat{\mathbf{A}}^+$ *is the generalised Moore-Penrose inverse of* $\widehat{\mathbf{A}}$.

Under these circumstance then, the marginal valuations may be recovered exactly for some, or possibly all, observations. If a full set of $\boldsymbol{\pi}_t$ vectors can be recovered, the condition (G) then becomes the quickest and most straightforward way to conduct the test. In any case the $\boldsymbol{\pi}_t$ vectors are likely to be of prime interest in any applied work.

So far we have assumed that prices are always observed. This is a very typical assumption but is rarely true in practice. A much more usual situation is one in which prices are observed for the goods which the consumer buys, but

[2] The computational problem is akin to that encountered in revealed preference tests of weak functional separability (see Varian (1983)).

[3] In the proof of Theorem 4 we define $\widetilde{\lambda}_t = \frac{\lambda_t}{\min\{\Lambda\}}$ and $\widetilde{\boldsymbol{\pi}}_t = \frac{\lambda_t \boldsymbol{\pi}_t}{\min\{\Lambda\}}$ where $\Lambda = \{\lambda_t\}_{t=1,\dots,T}$ and hence given feasible $\widetilde{\lambda}_t$ and $\widetilde{\boldsymbol{\pi}}_t$ terms we have a feasible $\boldsymbol{\pi}_t = \widetilde{\boldsymbol{\pi}}_t / \widetilde{\lambda}_t$.

not observed for those goods which he/she does not buy. In this situation it often becomes difficult or impossible[4] to test models nonparametrically. The specific problem lies in condition $(A.2)$ in Theorem 1. We now partition the price data and the corresponding rows of the $\mathbf{A}$ matrix in to two part. For the observed prices denoted with a superscript o (for observed, i.e. those for which $q_t^k > 0$) condition $(A.2)$ requires

$$\mathbf{p}_t^o = \mathbf{A}^o \boldsymbol{\pi}_t$$

whilst for the missing prices (denoted with a superscript m for missing)

$$\mathbf{p}_t^m \geq \mathbf{A}^m \boldsymbol{\pi}_t$$

If $\mathbf{p}_t^m$ is unobserved there is no way to implement this part of condition (A) in Theorem 1.

One possible procedure is to impute the missing values. The problem with such imputation is that if we then reject the characteristics model we are unsure whether the rejection is a genuine rejection or simply that we have the wrong imputed price. Instead we might prefer a procedure which searches for values for the missing prices so that the constructed data satisfy the necessary conditions. We have the following result

Theorem 7: *The following statements are equivalent.*
(U) *there exists prices* $\{\mathbf{p}_t^m\}_{t=1,...,T}$ *and a utility function* $v(\mathbf{z})$ *which is continuous and concave in characteristics and where* $\mathbf{z} = \mathbf{A}'\mathbf{q}$ *where the matrix* $\mathbf{A}$ *is known which z-rationalises the data* $\left\{[\mathbf{p}_t^o, \mathbf{p}_t^m]', \mathbf{q}_t, \mathbf{z}_t\right\}_{t=1,...,T}$.
(A) *there exist numbers and vectors* $\{V_t, \lambda_t > 0, \boldsymbol{\pi}_t, \mathbf{m}_t\}_{t=1,...,T}$ *such that*

$$\tilde{V}_s \leq \tilde{V}_t + \tilde{\boldsymbol{\pi}}_t' (\mathbf{z}_s - \mathbf{z}_t) \quad \textit{for all } s, t = 1, ..., T$$

$$\tilde{\lambda}_t \mathbf{p}_t^o \geq \mathbf{A}^o \tilde{\boldsymbol{\pi}}_t \quad \textit{for all } t = 1, ..., T$$

$$\mathbf{m}_t = \mathbf{A}^m \tilde{\boldsymbol{\pi}}_t \quad \textit{for all } t = 1, ..., T$$

This is clearly a weak test (which is just a special case of Theorem 6 if all the goods are bought) in the sense that the test on data in which all prices are observed (even if some quantities are zero) implies the conditions in Theorem 7, but not the other way around. The obvious interpretation of m_t^k is that it represents the marginal utility for good k in period t normalised by the marginal utility of income. In practice this linear programing approach, though still finite, may be very burdensome computationally.

4 Application

In this section we apply some of the ideas outlined above to data on purchases of (cow's) milk in a Danish consumer panel. The data cover 2,500 families during

[4] The problem arises whenever inner products like $\mathbf{p}_t'\mathbf{q}_s$ involve missing prices (for example, if the price of the Kth good is not observed in period t because it wasn't purchased, but is purchased in period s). Of course if a good is never bought it does not affect the test.

1999 and 2000. The data are monthly and report the quantities purchased and the prices paid for each type of milk. We concentrate on the 6 main types of milk (and drop buttermilk and flavoured milks) which differ according to fat content and conventional/organic production methods. Table 1 below shows the market shares by value and volume of these goods in our data. Overall conventional milk commanded the majority of the market with a share of about 75% on both measures, and semi-skimmed milk is the most popular type by fat content with nearly 60% of the market by both measures. This is mainly due to the pattern of conventional milk sales; within the organic segment of the market, skimmed and semi-skimmed have equal shares.

Table 1. Market shares

	Conventional milk			Organic milk		
Market share (%)	3.5% fat	1.5% fat	0.1% fat	3.5% fat	1.5% fat	0.1% fat
By value	14.68	45.79	14.16	4.00	10.72	10.64
By volume	13.65	49.26	15.54	3.03	9.16	9.36

Table 2 gives some descriptive statistics on the prices of the different types of milk in krona-per-litre. Organic milk is more expensive than conventionally produced milk by roughly 1.20 krona/litre, and within the organic/conventional split there is a clear gradient with respect to fat content: the higher the fat content the higher the price with a price difference of roughly 1krona/litre between high-fat and skimmed milk.

Table 2. Prices

	Conventional milk			Organic milk		
Prices (krona/litre)	3.5% fat	1.5% fat	0.1% fat	3.5% fat	1.5% fat	0.1% fat
Mean	6.11	5.34	5.09	7.34	6.51	6.30
Median	6.00	5.30	5.00	7.31	6.50	6.27
Std. Dev.	0.62	0.56	0.31	0.38	0.33	0.23
COV[5]	0.1007	0.1043	0.0614	0.0512	0.0509	0.0365

The characteristics model we have in mind for these data has $K = 6$ market goods, and $J = 3$ characteristics. These are listed in Table 3.

Table 3. Market goods and characteristics

Market goods	Characteristics
Conventional milk, 3.5% fat	"Milkiness"
Conventional milk, 1.5% fat	Fat content
Conventional milk, 0.1% fat	"Organic-ness"
Organic milk, 3.5% fat	
Organic milk, 1.5% fat	
Organic milk, 0.1% fat	
$K = 6$	$J = 3$

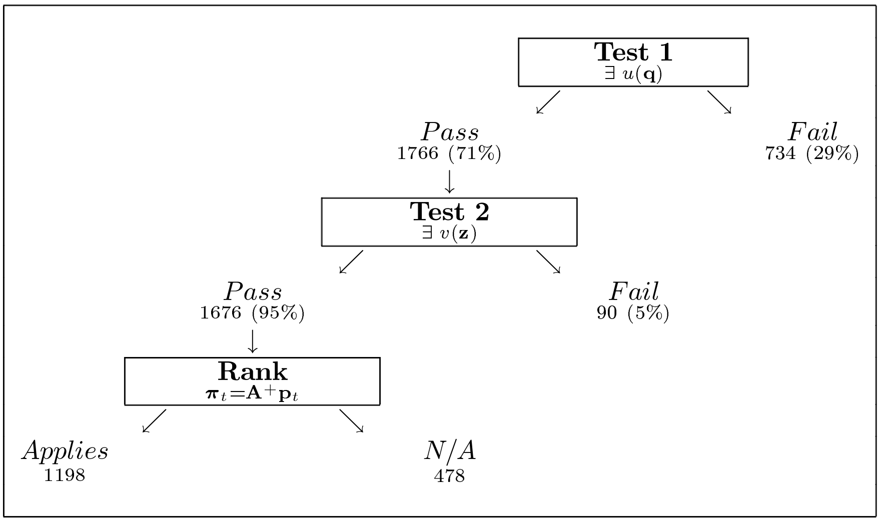
[5] Coefficient of variation

In what follows we test and apply the linear version of the characteristics model where

$$\mathbf{A} = \begin{bmatrix} \mathit{Milkiness} & \mathit{Fat} & \mathit{Organic} \\ 1 & 3.5 & 0 \\ 1 & 1.5 & 0 \\ 1 & 0.1 & 0 \\ 1 & 3.5 & 1 \\ 1 & 1.5 & 1 \\ 1 & 0.1 & 1 \end{bmatrix}$$

The first characteristic is is the baseline characteristic common to all products in this market – we have termed it "milkiness". Fat content is measured in cl's per litre and is linear with respect to market goods - if one buys two litres of full fat milk then it will contain 7cl's of fat by volume, a litre of skimmed and a litre of semi-skimmed together produce 1.6cl's of fat by volume. The interpretation of the organic/conventional characteristic is less straightforward. It is measured by an taking the value 1 if the organic characteristic is present, and 0 if it is not. Whenever the organic characteristic represents (perhaps a warm glow, perhaps the absence of antibiotic residues in the milk) this specification of the technology says that twice as much of it is produced by buying 2 litres of milk, than is produced by 1 litre – which is not to say, however, that the consumer then values it twice as much. This may not make complete sense, but if it is inconsistent with the data, then we should get a large number of rejections of the linear characteristics model, if not then this is a feature of an admissible characteristics model.

Table 4. Testing procedures and results



For each family in of our sample of 2500 we first test GARP on their individual $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\ldots,T}$ data to see whether there exists an admissible utility function $u(\mathbf{q})$ defined over products which rationalises their individual behaviour; for those where a suitable $u(\mathbf{q})$ exists we apply the linear programming condition in Theorem 1 to test whether their $\{\mathbf{p}_t, \mathbf{q}_t, \mathbf{z}_t\}_{t=1,\ldots,T}$ data can be rationalised with the rank-3 linear characteristics model describe above. Of those households whose behaviour does not reject this characteristics model and for whom an exact solution for their implied marginal valuations is possible we solve the simultaneous equation system involved and recover the $\boldsymbol{\pi}_t$ vectors. The testing procedure and the results are summarised in Table 4. Of the original sample of 2500 families, 1766 or 71% pass the GARP test and so we conclude that there exist utility functions which can q-rationalise their individual purchases of market goods, given the prices they face. For the remainder (29% of the sample), there exist no data-consistent utility functions defined over market goods, and since the characteristics model with $J < K$ is a special case of the general maximisation model, there is no characteristics-based utility function which can rationalise their individual behaviour either and we set them aside. For each of the remaining 1766 households, we run the linear programming test of the linear characteristics model. Of these households 1676 (95% of the surviving sample, or 67% of the original sample) satisfy the conditions so we know that there exist individual characteristics-conditioned utility functions which can rationalise their purchases of market goods and the rank-3 characteristics bundles. The 90 households which do not satisfy the condition (5% of the remaining sample) are set aside.

The linear program which provides the test of the characteristics model only indicates the existence of Afriat numbers and vectors of marginal valuations of characteristics which satisfy the conditions for the model; it does not recover them uniquely. For 71% of these households (1198 in total) unique solutions for the $\boldsymbol{\pi}-$vectors can be derived as these households are observed to buy multiple varieties of the market goods on at least one occasion. For these households we solve for $\boldsymbol{\pi}_t$ in each period where a solution can be obtained. Note that we may not be able to solve for all of the elements of $\boldsymbol{\pi}_t$ in every case as this will depend on the standard rank/order requirements for simultaneous equation systems. Overall we are able to recover characteristic valuations for 1198 households. For the milkiness characteristic we recover 6888 valuations, for fat 5113 valuations and for the organic characteristic 2281. Figure 2 illustrates the density of the distributions of these marginal valuations for each of the measured characteristics; milkiness, fat content and organic-ness. Table 5 reports some descriptive statistics for each characteristic.

Figure 1. The densities of the distributions of marginal valuations of characteristics

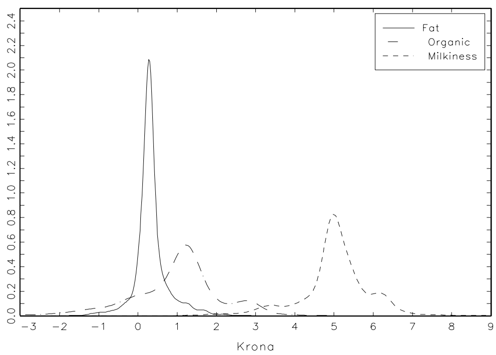
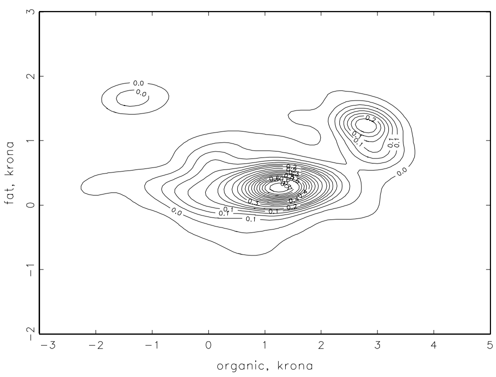


Table 5. Descriptive statistics of recovered marginal valuations, krona

	Percentile Points							
	Mean	10^{th}	25^{th}	50^{th}	75^{th}	90^{th}	Std err	n
Milkiness	5.02	3.86	4.80	5.00	5.44	6.15	1.01	6889
Fat	0.33	0.00	0.18	0.29	0.41	0.80	0.46	5687
Organic	0.99	-0.50	0.44	1.19	1.50	2.56	1.21	2744

On average milkiness has a valuation of just over 5 krona per litre with each additional cl of fat content increasing the valuation by 0.33 and organic attracting a premium of nearly 1 krona. The valuations of the milkiness and the organic characteristic are evidently highly heterogeneous, whilst the valuations of fat are less so. Milkiness always has a positive marginal valuation, however, there are some negative valuations for the other characteristics. In the case of fat 10% of valuations recovered are negative. In the case of organic 17% are negative. These negative marginal valuations do not conflict with the theory and are not inconsistent with the idea that the consumption of market goods has non-negative marginal utility overall. There are 1542 cases (involving 475 different households) in which we recover all three characteristic valuations simultaneously and using these we can construct a joint density of the distribution of (milkiness, fat, organic)[6]. Two dimensions (fat,organic) are illustrated in Figure 2 below.

Figure 2. The joint density of the distribution of marginal valuations of fat and organic



There appear to be three distinct groups of preferences: one small group (top left) which shows a high fat valuation, but a low valuation for organic; a larger group (top right) which has a high valuation of organic and a moderate one for fat; a large middle group which values organic but not fat. This figure illustrates "lumps" in the distribution of valuations and it is interesting to consider how this maps across to households. It seems that households are fairly stable with respect to which of these lumps they principally belong with 77% of households appearing in only one of these groups, 20% appearing in two groups (most often the big low fat-moderate organic and the top right moderate fat-moderate organic groups). Only 3% of households have valuations which appear in all three.

[6]The densities of the marginal distributions for this subset look virtually identical to those in Figure 1 except that the height of the density for fat is reduce.

5 Conclusions

We have extended the nonparametric methods developed in the literature to consumer characteristics models. We have derived the minimal necessary and sufficient empirical conditions under which data on the market behaviour of heterogeneous, price-taking, individual consumers are nonparametrically consistent with the consumer characteristics model. Where these conditions hold, we have show how information may be recovered on individual consumer's marginal valuations of product attributes. Where the conditions fail we highlight the role which the introduction of unobserved product attributes can play in rationalising the data. We have applied these ideas to consumer panel data on the Danish milk market.

Appendix: Proofs

Proof of Theorem 1

$(U) \Rightarrow (A)$: By concavity of $v(\mathbf{z})$ we have $v(\mathbf{z}_s) \leq v(\mathbf{z}_t) + \boldsymbol{\nabla} v(\mathbf{z}_t)' (\mathbf{z}_s - \mathbf{z}_t)$ and optimising behaviour implies $\mathbf{p}_t' \geq \boldsymbol{\pi}_t' \mathbf{A}'$ with equality when $q_t^k > 0$ and where $\lambda_t \boldsymbol{\pi}_t = \boldsymbol{\nabla} v(\mathbf{z}_t)$ by definition[7].

$(A) \Rightarrow (U)$: From condition $(A.2)$ we have $\mathbf{p}_t'\mathbf{q}_t = \boldsymbol{\pi}_t'\mathbf{A}'\mathbf{q}_t = \boldsymbol{\pi}_t'\mathbf{z}_t$, since the locations in $\mathbf{p}_t$ where the prices are greater than the corresponding locations in $\boldsymbol{\pi}_t'\mathbf{A}'$ occur only where the corresponding locations in $\mathbf{q}_t$ are zero[8]. The linear structure, however, ensures that $\mathbf{p}_t'\mathbf{q} \geq \boldsymbol{\pi}_t'\mathbf{A}'\mathbf{q} = \boldsymbol{\pi}_t'\mathbf{z}$. Hence $(A.2)$ gives $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\mathbf{q} \Rightarrow \boldsymbol{\pi}_t'\mathbf{z}_t \geq \boldsymbol{\pi}_t'\mathbf{z}$. Condition $(A.1)$ is, by Afriat's Theorem[9], equivalent to the existence of a concave, continuous utility function $v(\mathbf{z})$ such that for any $\mathbf{z}$ with $\boldsymbol{\pi}_t'\mathbf{z}_t \geq \boldsymbol{\pi}_t'\mathbf{z}$ it is the case that $v(\mathbf{z}_t) \geq v(\mathbf{z})$ (that is to say, condition $(A.1)$ means that there exists some $v(\mathbf{z})$ which q-rationalises the data $\{\boldsymbol{\pi}_t, \mathbf{z}_t\}_{t=1,\ldots,T}$). Combining these results we have that for any $\mathbf{z} = \mathbf{A}'\mathbf{q}$ such that $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\mathbf{q}$ then $\boldsymbol{\pi}_t'\mathbf{z}_t \geq \boldsymbol{\pi}_t'\mathbf{z}$ and there exists a suitable utility function $v(\mathbf{z})$ such that $v(\mathbf{z}_t) \geq v(\mathbf{z})$. Hence there exists a concave, continuous utility function $v(\mathbf{z})$ with $\mathbf{z} = \mathbf{A}'\mathbf{q}$ which z-rationalises the data $\{\mathbf{p}_t, \mathbf{q}_t, \mathbf{z}_t\}_{t=1,\ldots,T}$.

$(A) \iff (G)$: Given some $\boldsymbol{\pi}_t$ satisfying condition $(A.2)$, the equivalence between conditions $(A.1)$ and GARP on the data $\{\boldsymbol{\pi}_t, \mathbf{z}_t\}_{t=1,\ldots,T}$ follows from, for example, Theorem 1 in Varian (1983) by interpreting the $\boldsymbol{\pi}_t$ vector as the price vector for the characteristics.■

Proof of Corollary 1

If there exists a function $v(\mathbf{z})$ that z-rationalises $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\ldots,T}$, then $v(\mathbf{A}'\mathbf{q}_t) \geq v(\mathbf{A}'\mathbf{q})$ for all $\mathbf{q}$ such that $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\mathbf{q}$. Define $u(\mathbf{q})$ by $u(\mathbf{q}) = v(\mathbf{A}'\mathbf{q})$. Then $v(\mathbf{A}'\mathbf{q}_t) \geq v(\mathbf{A}'\mathbf{q})$ for all $\mathbf{q}$ such that $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\mathbf{q} \Longrightarrow u(\mathbf{q}_t) \geq u(\mathbf{q})$ for all $\mathbf{q}$ such that $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\mathbf{q}$. Therefore $u(\mathbf{q})$ q-rationalises the data. If $v(\mathbf{A}'\mathbf{q})$ is concave then $\mu v(\mathbf{A}'\mathbf{q}_t) + (1-\mu)v(\mathbf{A}'\mathbf{q}_s) = \mu u(\mathbf{q}_t) + (1-\mu)u(\mathbf{q}_s) \leq v(\mu\mathbf{A}'\mathbf{q}_t + (1-\mu)v\mathbf{A}'\mathbf{q}_s)$. But $v(\mu\mathbf{A}'\mathbf{q}_t + (1-\mu)v\mathbf{A}'\mathbf{q}_s) = v(\mathbf{A}'(\mu\mathbf{q}_t + (1-\mu)\mathbf{q}_s)) = u(\mu\mathbf{q}_t + (1-\mu)\mathbf{q}_s)$. Hence $\mu v(\mathbf{A}'\mathbf{q}_t) + (1-\mu)v(\mathbf{A}'\mathbf{q}_s) \leq u(\mu\mathbf{q}_t + (1-\mu)\mathbf{q}_s) \Longrightarrow u(\mathbf{q})$ is concave.■

Proof of Theorem 2

$(U) \Rightarrow (A)$: By concavity of $v(\mathbf{z})$ we have $v(\mathbf{z}_s) \leq v(\mathbf{z}_t) + \boldsymbol{\nabla} v(\mathbf{z}_t)' (\mathbf{z}_s - \mathbf{z}_t)$ and optimising behaviour implies $\mathbf{p}_t' \geq \boldsymbol{\pi}_t' \boldsymbol{\nabla}\mathbf{z}(\mathbf{q}_t)'$ with equality when $q_t^k > 0$ and where $\lambda_t \boldsymbol{\pi}_t = \boldsymbol{\nabla} v(\mathbf{z}_t)$ by definition. Now consider the concavity condition for the technology and pre-multiply by $\boldsymbol{\pi}_t \geq 0$

$$\begin{aligned} \mathbf{z}_s &\leq \mathbf{z}_t + \boldsymbol{\nabla}\mathbf{z}(\mathbf{q}_t)'(\mathbf{q}_s - \mathbf{q}_t) \\ &\Rightarrow \boldsymbol{\pi}_t'\mathbf{z}_s \leq \boldsymbol{\pi}_t'\mathbf{z}_t + \boldsymbol{\pi}_t'\boldsymbol{\nabla}\mathbf{z}(\mathbf{q}_t)'(\mathbf{q}_s - \mathbf{q}_t) \qquad \textit{for all } s,t = 1,...,T \end{aligned}$$

Substituting in $\mathbf{p}_t \geq \boldsymbol{\nabla}\mathbf{z}(\mathbf{q}_t)'\boldsymbol{\pi}_t$ preserves the inequality giving

$$\boldsymbol{\pi}_t'\mathbf{z}_s - \mathbf{p}_t'\mathbf{q}_s \leq \boldsymbol{\pi}_t'\mathbf{z}_t - \mathbf{p}_t'\mathbf{q}_t \qquad \textit{for all } s,t = 1,...,T$$

[7] Gorman (1956) p. 843.

[8] Linearity is over-sufficient in this sense as $\mathbf{p}_t'\mathbf{q}_t = \boldsymbol{\pi}_t'\mathbf{z}_t$ would also result from $\mathbf{z}(\mathbf{q})$ being a nonlinear homogenous of degree one function.

[9] Afriat (1967), Diewert (1973), Varian (1982).

which is the weak axiom of profit maximisation, WAPM (Varian (1984)). Finally, since WAPM must hold for all s given t stack the conditions horizontally and define

$$\mathbf{\Delta z}_t = [(\mathbf{z}_1 - \mathbf{z}_t), ..., (\mathbf{z}_T - \mathbf{z}_t)] \quad \mathbf{\Delta q}_t = [(\mathbf{q}_1 - \mathbf{q}_t), ..., (\mathbf{q}_T - \mathbf{q}_t)]$$

then we have condition $(A.2)$

$$\mathbf{\Delta z}_t' \boldsymbol{\pi}_t \leq \Delta \mathbf{q}_t' \mathbf{p}_t$$

$(A) \Rightarrow (U)$: From condition $(A.2)$ we have $\boldsymbol{\pi}_t'\mathbf{z}_t - \mathbf{p}_t'\mathbf{q}_t \geq \boldsymbol{\pi}_t'\mathbf{z}_s - \mathbf{p}_t'\mathbf{q}_s$. Hence $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\mathbf{q} \Rightarrow \boldsymbol{\pi}_t'\mathbf{z}_t \geq \boldsymbol{\pi}_t'\mathbf{z}$. The rest of the proof is analogous to that for Theorem 1.

$(A) \iff (G)$: Given some $\boldsymbol{\pi}_t$ satisfying condition $(A.2)$, the equivalence between conditions $(A.1)$ and GARP on the data $\{\boldsymbol{\pi}_t, \mathbf{z}_t\}_{t=1,...,T}$ follows from, for example, Theorem 1 in Varian (1983) by interpreting the $\boldsymbol{\pi}_t$ vector as the price vector for the characteristics.■

Proof of Corollary 2

Analogous to corollary 1.■

Proof of Lemma 1

See for example, Magnus and Neudecker, (1991), pp.36-37, Theorem 11. Note that uniqueness is guaranteed since $K < J$.■

Proof of Lemma 2

GARP for the data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,...,T}$ is equivalent to the existence of numbers $\{V_t, \lambda_t > 0\}_{t=1,...,T}$ such that

$$V_s \leq V_t + \lambda_t \mathbf{p}_t' (\mathbf{q}_s - \mathbf{q}_t) \qquad \textit{for all } s, t = 1, ..., T$$

By Lemma 1 the rank condition implies the existence of vectors $\{\boldsymbol{\pi}_t\}_{t=1,...,T}$ such that

$$\mathbf{p}_t' = \boldsymbol{\pi}_t' \mathbf{A}'$$

for all t. Substituting in means that there exists numbers $\{V_t, \lambda_t > 0\}_{t=1,...,T}$ such that

$$\begin{aligned} V_s &\leq V_t + \lambda_t \boldsymbol{\pi}_t' \mathbf{A}' (\mathbf{q}_s - \mathbf{q}_t) \qquad \textit{for all } s, t = 1, ..., T \\ V_s &\leq V_t + \lambda_t \boldsymbol{\pi}_t' (\mathbf{A}'\mathbf{q}_s - \mathbf{A}'\mathbf{q}_t) \qquad \textit{for all } s, t = 1, ..., T \\ V_s &\leq V_t + \lambda_t \boldsymbol{\pi}_t' (\mathbf{z}_s - \mathbf{z}_t) \qquad \textit{for all } s, t = 1, ..., T \end{aligned}$$

The rest of the proof is analogous to that for Theorem 1.■

Proof of Theorem 3

Let $\mathbf{P} = [\mathbf{p}_1, ..., \mathbf{p}_T]$ be $(K \times T)$. Let $\mathbf{\Pi} = [\boldsymbol{\pi}_1, ..., \boldsymbol{\pi}_t]$ be any $(J \times T)$ matrix such that $rank\{\mathbf{A} : \mathbf{E} : \mathbf{P}\} = rank\{\mathbf{A} : \mathbf{E}\}$ where

$$\mathbf{E} = \mathbf{P} - \mathbf{A\Pi}$$

As long as $\mathbf{\Pi}$ satisfies this rank condition it may be chosen arbitrarily – a straightforward way in which to guarantee the condition is to set $\mathbf{\Pi} = \mathbf{A}^{+}\mathbf{P}$ and then $\mathbf{E}\perp\mathbf{A}$. We have two equivalent formulations:

$$\mathbf{P} = [\mathbf{A} : \mathbf{E}] \begin{bmatrix} \mathbf{\Pi} \\ \mathbf{I}_T \end{bmatrix} \qquad \text{or} \qquad \mathbf{P} = [\mathbf{A} : \mathbf{I}_K] \begin{bmatrix} \mathbf{\Pi} \\ \mathbf{E} \end{bmatrix}$$

and in both cases we have

$$rank\left\{\overline{\mathbf{A}} \ : \mathbf{P}\right\} = \ rank\left\{\overline{\mathbf{A}}\right\}$$

where $\overline{\mathbf{A}}$ is the augmented $\mathbf{A}$ matrix in either case. We can therefore add the lesser of K or T new characteristics, and given the rank condition, Lemma 1 applies and we can z-rationalise these "data".■

Proof of Theorem 4

(Sufficiency) Let $J < K < T$. If $\text{rank}\{\mathbf{P}\} = J < K$ then we can find a $J \times K$ column basis matrix for $\mathbf{P}$. Denote this column basis by $\mathbf{A}'$. By definition $\mathbf{A}'$ (and therefore $\mathbf{A}$) has rank J and $\text{rank}\{\mathbf{A} : \mathbf{P}\} = \text{rank}\{\mathbf{A}\}$. Therefore $\text{rank}\{\mathbf{A} : \mathbf{p}_t\} = \text{rank}\{\mathbf{A}'\}$ for all t. If in addition the data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\ldots,T}$ pass GARP then Lemma 2 implies that $v\left(\mathbf{A}'\mathbf{q}\right)$ z-rationalises the data $\{\mathbf{p}_t, \mathbf{q}_t, \mathbf{z}_t\}_{t=1,\ldots,T}$ *where* $\mathbf{z}_t = \mathbf{A}'\mathbf{q}_t$.
(Necessity) If all market goods are bought (or are on the verge of being bought) and there exists a non-trivial linear characteristics model such that $v\left(\mathbf{A}'\mathbf{q}\right)$ z-rationalises the data $\{\mathbf{p}_t, \mathbf{q}_t, \mathbf{z}_t\}_{t=1,\ldots,T}$ where $\mathbf{z}_t = \mathbf{A}'\mathbf{q}_t$ then Lemma 2 implies that $\text{rank}\{\mathbf{A} : \mathbf{p}_t\} = \text{rank}\{\mathbf{A}\}$ for all t and that the data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\ldots,T}$ pass GARP. If $\text{rank}\{\mathbf{A} : \mathbf{p}_t\} = \text{rank}\{\mathbf{A}\}$ for all t then $\text{rank}\{\mathbf{A} : \mathbf{p}_t\} = \text{rank}\{\mathbf{A}\} = J$ and hence $\text{rank}\{\mathbf{P}\} \leq J < K$.

Note that, since we choose a column basis of $\mathbf{P}$ for $\mathbf{A}$, and every element of $\mathbf{P}$ is non-negative, we can always choose a basis and therefore an $\mathbf{A}$ which has every element non-negative by choosing linearly independent columns of $\mathbf{P}$ to be the basis. Also, consider the set of equations $\mathbf{P} = \mathbf{A\Pi}$ where Π is an unknown $J \times T$ matrix. By the definition of $\mathbf{A}$ $\text{rank}\{\mathbf{A} : \mathbf{P}\} = \text{rank}\{\mathbf{A}\}$ and thus, by Lemma 1, a solution for $\mathbf{\Pi}$ exists.■

Proof of Theorem 5

Let $\Lambda = \{\lambda_t\}_{t=1,\ldots,T}$ for a set of strictly positive constants which satisfy condition (A) in Theorem 1. To see that condition (A) in Theorem 1 can be rewritten as condition (A) in Theorem 4, let $\min\{\Lambda\}$ be the smallest element of this set and let $\widetilde{V}_t = \frac{V_t}{\min\{\Lambda\}}$, $\widetilde{\lambda}_t = \frac{\lambda_t}{\min\{\Lambda\}}$ and $\widetilde{\boldsymbol{\pi}}_t = \frac{\lambda_t \boldsymbol{\pi}_t}{\min\{\Lambda\}}$. Then condition (A) in Theorem 1 can be rewritten

$$\widetilde{V}_s \leq \widetilde{V}_t + \widetilde{\boldsymbol{\pi}}_t'\left(\mathbf{z}_s - \mathbf{z}_t\right) \qquad \text{for all } s, t = 1, ..., T$$

and

$$\tilde{\lambda}_t \mathbf{p}_t \geq \mathbf{A}\widetilde{\boldsymbol{\pi}}_t \qquad \text{with equality when } q_t^k > 0 \text{ for all } t = 1, ..., T$$

where $\widetilde{\lambda}_t \geq 1$. Now suppose we have $\left\{\widetilde{V}_t, \widetilde{\lambda}_t, \widetilde{\boldsymbol{\pi}}_t\right\}$ which satisfy these inequalities. Dividing by the positive constant $\widetilde{\lambda}_t$ preserves the first inequality gives

$$\frac{\widetilde{V}_s}{\widetilde{\lambda}_t} \leq \frac{\widetilde{V}_t}{\widetilde{\lambda}_t} + \frac{1}{\widetilde{\lambda}_t}\widetilde{\boldsymbol{\pi}}_t'\left(\mathbf{z}_s - \mathbf{z}_t\right)$$

and

$$\mathbf{p}_t \geq \mathbf{A}\tilde{\boldsymbol{\pi}}_t \frac{1}{\tilde{\lambda}_t}$$

and defining $V_t = \frac{\tilde{V}_t}{\tilde{\lambda}_t}$, $\lambda_t = \frac{1}{\tilde{\lambda}_t}$ and $\boldsymbol{\pi}_t = \tilde{\boldsymbol{\pi}}_t \frac{1}{\tilde{\lambda}_t}$ we have condition (A) in Theorem 1. ■

Proof of Theorem 6

Follows immediately from Lemma 1.■

Proof of Theorem 7

Consider the condition in Theorem 6, partitioned according to whether the price data are observed (i.e. $q_t^k > 0$) or not.

$$\tilde{V}_s \leq \tilde{V}_t + \tilde{\boldsymbol{\pi}}_t' (\mathbf{z}_s - \mathbf{z}_t) \quad \textit{for all } s, t = 1, ..., T$$
$$\tilde{\lambda}_t \mathbf{p}_t^o = \mathbf{A}^o \tilde{\boldsymbol{\pi}}_t \quad \textit{for all } t = 1, ..., T$$
$$\tilde{\lambda}_t \mathbf{p}_t^m \geq \mathbf{A}^m \tilde{\boldsymbol{\pi}}_t \quad \textit{for all } t = 1, ..., T$$

This condition is equivalent to condition (A) in Theorem 1 however since $\mathbf{p}_t^m$ is not observed this is no longer a linear programme. Instead we combine the two unknowns, $\tilde{\lambda}_t$ and $\mathbf{p}_t^m$, into the vector $\mathbf{m}_t$ and we have the condition in Theorem 7. This is now in linear programming form. From Theorems 1 and 6, if a suitable $\mathbf{p}_t^m$ exists then this linear programme is feasible and we can find suitable $\mathbf{m}_t$ vectors. Conversely if this linear programme has a feasible solution then we can construct suitable $\mathbf{p}_t^m$ vectors since $\mathbf{p}_t^m = \frac{\mathbf{m}_t}{\tilde{\lambda}_t}$. ■

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