Productivity Measurement under Uncertainty

Robert G. Chambers[1]

University of Maryland and University of Western Australia

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Abstract

Stochastic productivity indicators are defined, and superlative measures of these indicators are derived for risk-neutral producers. It is shown that in a rational-expectations equilibrium, differences in the market values of firms are superlative indicators of cross-sectional productivity differences. An illustrative regression-based method for estimating the intertemporal superlative productivity indicators is discussed and applied to data for postwar US agriculture. Comparison with existing nonstochastic productivity indicators suggests that properly accounting for the stochastic environment in which firms operate could have important empirical implications for measuring productivity growth.

Key words: productivity measurement, uncertainty, risk

Productivity Measurement under Uncertainty

Most currently computed economic productivity indexes are based on the specification of a primal productivity index, the assumption of price-taking behavior on the part of producers, and superlative approximation results originally due to Diewert (1976). Caves, Christensen, and Diewert's (1982, 1982) classic derivations of the Törnqvist productivity index from a transcendental logarithmic radial distance function define and epitomize this approach. In deriving these indexes, the basic tactic is to reduce a productivity index specified in primal terms to a differential approximation and then to use the assumption of price-taking behavior to obtain easily calculable measures.[1] A salient assumption in these approximation arguments is that both the primal technology and the decision environment are nonstochastic.

This paper develops productivity measures for producers facing stochastic technologies and a stochastic decision environment. Superlative approximations to these stochastic productivity measures are then derived for risk-neutral producers. Among other results, it is shown that, in a rational expectations equilibrium, differences in firm values provide superlative measures of productivity differences.

If one is to consider stochastic productivity measures seriously, some *prima facie* evidence should exist that current productivity measures, which ignore the stochastic nature of technology and the decision environment, are potentially misleading. Even though it is a truism that the world is a very uncertain place, uncertainty may not affect practical productivity measurement. To this end, Figure 1 portrays the logarithmic differences of the official total factor productivity measure for US agriculture reported by the United States Department of Agriculture (USDA) for the years 1949 to 1999.[2] These differences approximate year-to-year percentage changes in measured US agricultural total factor productivity. Through the 1950s and 1960s, measured productivity seemed to grow at a reasonably stable

[1] Diewert (1987) and Balk (1998) contain summaries of the state of the art in the economic computation of index numbers.

[2] USDA was the first federal agency to introduce multifactor productivity measures into its statistical program starting in 1960. The total factor productivity measure from which this logarithmic difference is derived as well as the data underlying the calculation of the total factor productivity measure are publicly available at www.ers.usda.gov/data/AgProductivity. This measure is also reported annually in the data appendix to the *Economic Report of the President.*

rate with only intermittent declines. Beginning in the 1970s, however, the pattern of change became increasingly unstable. Relatively sharp falls in total factor productivity are followed year-on-year by relatively sharp rises in total factor productivity. In the early 1980s, what can be only be described as wildly erratic changes in total factor productivity occur. Percentage declines approaching 15% are followed by percentage increases exceeding 15%. The wide swings dampened down for a bit, but in the 1990s, the oscillations returned.

Ex post, it is easy to explain virtually all of this variability. In particular, the dramatic fluctuation in the 1980s can be partially attributed to severe drought conditions in the United States. Similar explanations exist for the 1970s and the 1990s. After all, agriculture is the prototypical stochastic production process. Dramatic and even cataclysmic drops in output due to poor weather or unforeseen market events, in some sense, are perceived as the norm and not as the exception.

But truth be told, these are actually explanations why output may vary widely for a given utilization of measured inputs for a technology that is inherently stochastic. Is it necessarily true that stochastic technologies exhibit variable productivity growth? If one adopts as their definition of total factor productivity, as is traditional, *some* index of *measured output* divided by *some* index of *measured inputs,* the answer is perhaps yes. The essence of the economic approach to productivity measurement, however, is that measured changes in productivity should capture more than that. In particular, the literature on exact and superlative indexes argues that productivity measures reflect differences in production frontiers (suitably defined) across either time or production units. The USDA total factor productivity measures are calculated under just such a premise.

Is it plausible that US agriculture as a sector (or any other industry for that matter) really experiences year-on-year shifts in its production frontier on the magnitude of 15% in different directions? That seems far-fetched. If true, it immediately brings to mind the old saw that "...productivity only measures our ignorance". Moreover, in an era when unexpectedly slow rates of productivity growth can merit headline coverage in the financial pages of most leading newspapers, such a view potentially leaves one of the more important indicators of the future health of the economy squarely within the provenance of chance.

The first premise of this paper is that productivity measures should accurately account for the stochastic world in which decisionmakers operate. The real world is a very risky and uncertain place. The ultimate goal should be measures that can discriminate between true losses in productive ability and just plain 'luck', be it good or bad. Developing such measures requires that the stochastic decision environment producers face be modelled from the beginning. The second premise of this paper is that productivity measures for stochastic technologies should be recognizable generalizations of productivity measures for nonstochastic technologies, and not simple *ex post* corrections to or rationalizations of nonstochastic measures. Or put another way, nonstochastic productivity measurement should be a special case of stochastic productivity measurement. The third premise is that the stochastic productivity measure developed should be relatively easy to compute and should not require detailed knowledge of the underlying technology. The economic approach to productivity measurement teaches us that if one *knows* the production technology, then one can directly derive productivity measures from it. The use of superlative index numbers is predicated upon the realization that all too frequently the technology is not known or feasibly estimable. Thus, it would be good if stochastic productivity measures exist that do not require either exact or statistically precise knowledge of the technology.

In what follows, the basic model is first specified. Then a primal economic productivity index for a stochastic technology is defined. After the indicator is specified in primal terms, exact measures are derived for a flexible representation of the stochastic primal technology. The resulting exact measures are superlative indicators in the sense of Diewert (1976). As noted, a particularly striking result is the demonstration that the differences in the market values of two firms is a superlative indicator of their productivity differences in a rational-expectations equilibrium. The penultimate section contains an empirical application of some of the theoretical results to intertemporal productivity measurement. While the analysis here is preliminary, it seems to indicate that properly accounting for the stochastic nature of technology can play an important role in empirical productivity measurement. The final section concludes.

1 The Model

Firms face an uncertain or risky environment in a two-period setting. The current period, 0, is certain, but the future period, 1, is uncertain. Uncertainty or risk is resolved by 'Nature' making a choice from $\Omega = \{1, 2, ..., S\}$.[3] Each element of Ω is referred to as a state of nature.

Firms have two ways in which to convert period 0 income or investments into period 1 stochastic income. The first is a state-contingent production technology, which is represented by a single-product, state-contingent input correspondence.[4] To make this explicit, let $\mathbf{x} \in \Re_{+}^{N}$ be a vector of inputs committed prior to the resolution of uncertainty (period 0), and let $\mathbf{z} \in \Re_{+}^{S}$ be a vector of *ex ante* or state-contingent outputs also chosen in period 0.[5] If state $s \in \Omega$ is realized (picked by 'Nature'), and the producer has chosen the *ex ante* input-output combination $(\mathbf{x}, \mathbf{z})$, then the realized or *ex post* output in period 1 is z_s. The period 1 price of the product of this stochastic technology is also stochastic and denoted by the state-contingent price vector $\mathbf{p} \in \Re_{++}^{S}$.

The technology is characterized by a continuous input correspondence, $X : \Re_{+}^{S} \rightarrow \Re_{+}^{N}$, which maps state-contingent output vectors into input sets that are capable of producing that state-contingent output vector.[6] It is defined

$$X(\mathbf{z}) = \{\mathbf{x} \in \Re_{+}^{N} : \mathbf{x} \text{ can produce } \mathbf{z}\}.$$

Intuitively, $X(\mathbf{z})$ is associated with everything on or above the isoquant for the state-contingent output vector $\mathbf{z}$. I impose the following properties on $X(\mathbf{z})$:

X.1 $\mathbf{z}' \leq \mathbf{z} \Rightarrow X(\mathbf{z}) \subseteq X(\mathbf{z}')$.

[3] The assumption of a finite state space is made in the theoretical section to facilitate comparison with multiproduct productivity indicators. It is not essential to analysis.

[4] Our application is to a single product technology. The technology is, therefore, assumed to produce a single stochastic product to simplify the notation. It is trivial to follow the arguments of Chambers and Quiggin (2000) and Chambers (2002) to deduce the multiple output analogues to our results. Ultimately, the superlative indicators derived correspond exactly to those below except that they involve summation over both outputs and states of Nature.

[5] Once Ω is endowed with a probability measure, $\Re^{\Omega} = \Re^{S}$ can be interpreted as the space of random variables.

[6] Different firms are presumed to face different technologies. The technology for firm n will be represented by a correspondence X^n. I suppress the n where there can be no confusion.

X.2 $\mathbf{x}' \geq \mathbf{x} \in X(\mathbf{z}) \Rightarrow \mathbf{x}' \in X(\mathbf{z})$.

X.3 X is continuous.

X.1 says that if an input combination can produce a particular mix of state-contingent outputs then it can always be used to produce a smaller mix of state-contingent outputs. X.2 implies that inputs have non-negative marginal productivity.

Period 0 prices of inputs are denoted by $\mathbf{w} \in \Re^N_{++}$ and are nonstochastic.

Firms can also produce income in period 1 from period 0 expenditures via investment in frictionless financial markets.[7] The *ex ante* financial security payoffs are given by the $S \times J$ non-negative matrix $\mathbf{A}$. The period 0 prices of the financial securities are given by $\mathbf{v} \in \Re^J_+$. The firm's portfolio vector is denoted $\mathbf{h} \in \Re^J$ so that its period 1 payout from its financial position is $\mathbf{Ah} \in \Re^S$.

The exogenous period 0 wealth level is w_0 while the firm's period 1 wealth endowment is normalized with no loss of generality at zero. The firm's preferences over consumption in period 0, c_0, and period 1 stochastic consumption, $\mathbf{c} \in \Re^S_+$, assume the risk-neutral form

$$\delta \boldsymbol{\pi}' \mathbf{c} + c_0$$

where the discount factor $\delta < 1$ discounts period 1 consumption back to period 0, and $\boldsymbol{\pi}' = (\pi_1, ..., \pi_S)$ is a vector of subjective probabilities.[8]

The firm's problem, therefore, is

$$\max_{\mathbf{x},\mathbf{z},\mathbf{h}} \{w_0 - \mathbf{w}'\mathbf{x} - \mathbf{v}'\mathbf{h} + \delta \boldsymbol{\pi}' \mathbf{c} : \mathbf{x} \in X(\mathbf{z}), \mathbf{Ah} + \mathbf{p} \bullet \mathbf{z} \geq \mathbf{c}\}, \quad (1)$$

[7] Identical productivity measures emerge for the risk-neutral firm regardless of whether it has access to competitive financial markets or not. This happens because the assumption of risk-neutrality parameterizes the firm's internal state-claim prices as its probabilities. However, if the firm is not risk neutral then the firm's access to financial markets is an important determinant of its endogenous internal state-claim prices. Thus, in the more general case, final productivity measures depend importantly on the firm's access to financial markets (Chambers, 2003). For the sake of completeness and ultimate comparability with the more general case, we allow the firm to have access to financial markets even though it does affect the actual productivity measure derived.

[8] The preference structure imposes both risk neutrality and constancy of the rate of time preference across nonstochastic consumption levels. The latter can be relaxed with almost no change in the analysis. Moreover, if the the asset structure $\mathbf{A}$ contains a riskless asset, there is no real loss of generality in setting a constant discount factor.

where the notation $\mathbf{p} \bullet \mathbf{z}$ denotes the elementwise product $(p_1 z_1, ..., p_S z_S)$.

2 A Primal Productivity Indicator for Stochastic Technologies

The most common approach to specifying an economic productivity index is to base productivity comparisons on ratios, more precisely logarithmic averages of ratios, of radial distance functions. Caves, Christensen, and Diewert (1982) refer to these measures as Malmquist productivity indexes.

More recently, Chambers (1996, 1998, 2002) has defined a series of primal productivity indicators based on differences between directional distance functions. As Chambers, Chung, and Färe (1996) show, directional distance functions contain the more familiar radial distance functions as interesting special cases. Chambers has referred to the resulting difference-based productivity measures as *Luenberger productivity indicators.*

Here I consider state-contingent versions of a Luenberger input-based productivity indicator. This proves an especially convenient analytic choice because it allows one to deduce measures of these indicators in value terms that are normalized by variables (the value of the reference bundle of inputs) exogenous to the individual.[9] The resulting indicators are also very convenient intuitively because, as demonstrated in the next section, they are superlatively measured by Bennet-Bowley indicators. These Bennet-Bowley indicators, in turn, clarify the link between measures of productivity and the market values of firms in a rational expectations equilibrium. In any case, the style of argument, which follows well-established techniques originally due to Caves, Christensen, and Diewert (1982), varies little across the type of indicator chosen. Thus, once the general principles are set, the arguments can be readily extended to other measures.[10] Moreover, as I show below, computed values of the

[9] For the risk-neutral firm, this advantage is slightly diminished if there exist objective probabilities with which to discount stochastic output. However, if the firm is not risk neutral, then its intertemporal rate of substitution is both stochastic and endogeneous to the individual. Hence, it is very inconvenient for normalization purposes. Measures based on radial output distance functions can suffer a similar shortcoming.

[10] In particular, arguments directly parallel to the arguments used here can be applied to radial productivity measures such as the Malmquist productivity index developed by Caves, Christensen, and Diewert (1982,

nonstochastic version of the productivity indicator derived here closely match the currently computed radial based measures for my data set.[11]

For $\mathbf{g} \in \Re_{+}^{N}$, define the directional input distance function for $X(\mathbf{z})$ by

$$D(\mathbf{x}, \mathbf{z}; \mathbf{g}) = \max \{\beta \in \Re : \mathbf{x} - \beta \mathbf{g} \in X(\mathbf{z})\},$$

if there is some β such that $\mathbf{x} - \beta \mathbf{g} \in X(\mathbf{z})$ and $-\infty$ otherwise. In words, $D(\mathbf{x}, \mathbf{z}; \mathbf{g})$ represents the maximal number of units of a reference input vector, $\mathbf{g}$, that can be subtracted from $\mathbf{x}$ and still leave the resulting input vector, $\mathbf{x} - \beta \mathbf{g}$, capable of producing $\mathbf{z}$. Geometrically, it can be visualized as measuring the distance that $\mathbf{x}$ can be projected in the direction of $\mathbf{g}$ before it hits the isoquant for the input set $X(\mathbf{z})$.

Given properties X for the input set, D satisfies (Luenberger, 1992; Chambers, Chung, and Färe, 1996; Chambers and Quiggin, 2000):

D.1 $D(\mathbf{x} - \alpha \mathbf{g}, \mathbf{z}; \mathbf{g}) = D(\mathbf{x}, \mathbf{z}; \mathbf{g}) - \alpha, \alpha \in \Re$;

D.2: D is upper semi-continuous in $\mathbf{x}$ and $\mathbf{z}$ (jointly);

D.3: $D(\mathbf{x}, \mathbf{z}; \lambda \mathbf{g}) = \frac{1}{\lambda} D(\mathbf{x}, \mathbf{y}; \mathbf{g}), \lambda > 0$;

D.4: $(\mathbf{x}', -\mathbf{z}') \geq (\mathbf{x}, -\mathbf{z}) \Rightarrow D(\mathbf{x}', \mathbf{z}'; \mathbf{g}) \geq D(\mathbf{x}, \mathbf{z}; \mathbf{g})$, i.e., nondecreasing in inputs and nonincreasing in output;

D.5: $\mathbf{x} \in X(\mathbf{z}) \Leftrightarrow D(\mathbf{x}, \mathbf{y}; \mathbf{g}) \geq 0$.

The stochastic Luenberger input-based productivity indicator for firms indexed by $(0, 1)$ is defined

$$L(\mathbf{x}^1, \mathbf{x}^0, \mathbf{z}^1, \mathbf{z}^0) = \frac{1}{2} \left[D^1(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g}) - D^1(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g}) + D^0(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g}) - D^0(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g}) \right], \quad (2)$$

where superscripts now index either different firms in a cross section or the same firm in different time periods.

To understand why $L(\mathbf{x}^1, \mathbf{x}^0, \mathbf{z}^1, \mathbf{z}^0)$ is a reasonable measure of productivity differences across firms, consider each of its components. $D^1(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g})$ is the maximum number of units of the reference vector, $\mathbf{g}$, that can be subtracted from $\mathbf{x}^0$ and leave the resulting input vector in $X^1(\mathbf{z}^0)$, which is the input set for firm $1's$ technology but firm $0's$ state-contingent output

1983). I leave the actual manipulations to the interested reader.

[11] Advantages and disadvantages of this type of difference-based productivity indicator are discussed more completely in Chambers (1998, 2002), Balk (1998), Diewert (1998), and Fox (2003).

distribution, $\mathbf{z}^0$. If $D^1(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g}) > 0$, then a smaller input bundle than $\mathbf{x}^0$ can produce $\mathbf{z}^0$ using firm 1's technology. Hence, firm 0 would realize a cost saving over $\mathbf{x}^0$ in producing $\mathbf{z}^0$ if it had access to firm $1's$ technology. The second term, $D^1(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g})$, is the maximum number of units of $\mathbf{g}$ that can be subtracted from $\mathbf{x}^1$ and still leave the translated input in $X^1(\mathbf{z}^1)$. Because optimizing firms always operate on the frontier of their technology, $D^1(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g}) = 0$ in what follows. Hence, the first difference, $D^1(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g}) - D^1(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g})$, measures (in units of the reference input vector) potential cost savings to firm 0 that would be available if it had access to firm $1's$ technology. If $D^1(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g}) - D^1(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g}) > 0$, then firm $1's$ technology is reasonably said to be more productive than firm $0's$ because firm 0 would realize a cost savings if it used firm 1's technology in place of its own.

Similarly, $D^0(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g}) - D^0(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g})$ measures how many units of $\mathbf{g}$ that firm 1 can save by having access to firm $0's$ technology. Because it is possible, for example, to have $D^1(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g}) - D^1(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g}) > 0$ but $D^0(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g}) - D^0(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g}) < 0$, the Luenberger productivity indicator averages these terms in arriving at the overall productivity indicator.[12] If $L(\mathbf{x}^1, \mathbf{x}^0, \mathbf{z}^1, \mathbf{z}^0) > 0$, firm 1 is judged to be more productive than firm 0.

The stochastic Luenberger productivity indicator generalizes analogous nonstochastic productivity indicators defined in Chambers (1996, 2002) to state-contingent technologies. It is also straightforward to define Malmquist stochastic productivity indexes, which generalize the familiar nonstochastic Malmquist productivity indexes, in an exactly analogous fashion using radial distance functions defined on the state-contingent technology characterized by $X(\mathbf{z})$. That is left to the interested reader.

While straightforward once production and decision uncertainty are modelled in an Arrow-Debreu state-space set up, the generalization remains interesting for several reasons. Most importantly, it forces the recognition that primal productivity indexes or indicators for stochastic technologies properly depend upon the *entire output distribution* of both firms being compared. Stochastic productivity measurement inherently involves measures that summarize the stochastic distribution of outputs across Ω. Thus, almost by definition, stochastic productivity measurement inherently involves statistical analysis.

[12] For similar reasons, Malmquist indexes rely on geometric averages of the analogous ratio-based measures (Caves, Christensen, and Diewert).

This recognition creates practical difficulties for stochastic productivity measurement that do not confront nonstochastic productivity measurement. To illustrate, suppose that one actually knew D^0 and D^1. In the nonstochastic case, this knowledge would permit direct calculation of a productivity index or indicator if one had observations on inputs and outputs. There would, in fact, be no need for exact approximation results or superlative indexes. In short, the economic approach to productivity index construction would be redundant.

In the stochastic case, however, *the productivity indicator remains uncomputable even with exact knowledge about the technology.* As shown above, the indicator properly depends not only upon observed inputs but also upon both stochastic output distributions in their entirety. Data are simply not available on *ex ante* output distributions. Instead, one only has data on realized or *ex post* (and thus observable) output. To compute the productivity indicator, one must somehow use the *ex post* output data to infer the structure of the *ex ante* output choice. This entails using sample data to infer relevant information about the underlying output distribution. That is the natural domain of statistics.

Thus, the search for computable versions of these indicators reduces to a search for convenient summary representations (i.e., statistics) of these distributions. The firm's attitudes towards risk play a fundamental role in developing such measures. The less the firm cares about risk, the more straightforward will be the task in assembling and computing such measures. Here we study the polar case, where the firm is indifferent to risk, and as consequence we are able to develop extremely straightforward measures.

3 Superlative Stochastic Productivity Indicators

Following the approach of Caves, Christensen, and Diewert (1982, 1982), the first step in calculating a superlative productivity indicator is to specify a flexible functional form that provides a second-order differential approximation to a complete function representation of the technology. The form chosen here is the following quadratic approximation to the directional distance function defined above for firm h

$$D^h(\mathbf{x},\mathbf{z};\mathbf{g}) = a_0^h + \sum_{i=1}^{N} a_i^h x_i + \sum_{k=1}^{S} b_k^h z_k + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_{ij}x_i x_j + \frac{1}{2}\sum_{k=1}^{S}\sum_{l=1}^{S}\beta_{kl}z_k z_l + \sum_{i=1}^{N}\sum_{k=1}^{S}\gamma_{ik}x_i z_k, \tag{3}$$

with

$$\begin{aligned}\alpha_{ij} &= \alpha_{ji}, \beta_{kl} = \beta_{lk}, \sum_{i=1}^{N} a_i^h g_i = 1; \\ \sum_{j=1}^{N} \alpha_{ij} g_j &= 0, i = 1, ..., N; \sum_{i=1}^{N} \gamma_{ik} g_i = 0, k = 1, ..., S.\end{aligned}$$

Chambers (2001) has shown that the quadratic directional distance function (3) is a second-order flexible approximation to any twice differentiable directional input distance. Notice that, for this specification, D^0 and D^1 differ only in their first-order or linear terms. Superlative approximations to Malmquist productivity measures rely on similar assumptions about technical differences across firms.

The next step in calculating a superlative indicator is to reduce (2) for this flexible representation of the technology to something that is potentially calculable using relatively easily observable data. The key to this decomposition is an approximation result drawn from Diewert's (1976) classic paper on index numbers. In what follows, ∇ denotes the gradient with respect to the subscripted vector.

Lemma 1 *(Diewert) $f(\mathbf{q})$ is expressible as*

$$a_0 + \sum_{i=1}^{n} a_i q_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} q_i q_k$$

$b_{ik} = b_{ki,}$ *if and only if:*

$$f(\mathbf{q}^1) - f(\mathbf{q}^0) = \frac{1}{2} \left[\nabla_z f(\mathbf{q}^1) + \nabla_z f(\mathbf{q}^0)\right]' \left(\mathbf{q}^1 - \mathbf{q}^0\right)$$

Applying Lemma 1 to (3), we obtain:

$$\begin{aligned}L\left(\mathbf{x}^1, \mathbf{x}^0, \mathbf{z}^1, \mathbf{z}^0\right) &= \frac{1}{2} \left[\nabla_{\mathbf{x}} D^1\left(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g}\right) + \nabla_{\mathbf{x}} D^0\left(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g}\right)\right]' \left[\mathbf{x}^0 - \mathbf{x}^1\right] + \\ &\quad \frac{1}{2} \left[\nabla_{\mathbf{z}} D^1\left(\mathbf{x}^1, \mathbf{z}^1; \mathbf{g}\right) + \nabla_{\mathbf{z}} D^0\left(\mathbf{x}^0, \mathbf{z}^0; \mathbf{g}\right)\right]' \left[\mathbf{z}^0 - \mathbf{z}^1\right].\end{aligned} \tag{4}$$

The final step is to use the firm's optimization behavior to express (4) in something close to observable form. This is accomplished by using D.1, D.5, and following Chambers (2001) to show that problem (1) can be rewritten

$$\max_{\mathbf{x},\mathbf{z},\mathbf{h}}\left\{w_0-\mathbf{w}'\left(\mathbf{x}-D^h\left(\mathbf{x}^h,\mathbf{z}^h;\mathbf{g}\right)\mathbf{g}\right)-\mathbf{vh}+\delta\boldsymbol{\pi}'\mathbf{c}:\mathbf{Ah}+\mathbf{p}\bullet\mathbf{z}\geq\mathbf{c}\right\}.$$

The first-order conditions for the reformulated problem are

$$\begin{aligned}\nabla_{\mathbf{x}}D\left(\mathbf{x},\mathbf{z};\mathbf{g}\right)\left(\mathbf{w}'\mathbf{g}\right)-\mathbf{w} &\leq \mathbf{0}, \quad \mathbf{x}\geq\mathbf{0}\\ \nabla_{\mathbf{z}}D\left(\mathbf{x},\mathbf{z};\mathbf{g}\right)\left(\mathbf{w}'\mathbf{g}\right)+\boldsymbol{\mu}\bullet\mathbf{p} &\leq \mathbf{0}, \quad \mathbf{z}\geq\mathbf{0}\\ \delta\boldsymbol{\pi}-\boldsymbol{\mu} &\leq \mathbf{0}, \quad \mathbf{c}\geq\mathbf{0},\\ -\mathbf{v}+\boldsymbol{\mu}'\mathbf{A} &= \mathbf{0},\end{aligned}$$

in the notation of complementary slackness, where $\boldsymbol{\mu}'=(\mu_1,...,\mu_S)$ is a row vector of non-negative Lagrange multipliers associated with the S constraints $\mathbf{Ah}+\mathbf{p}\bullet\mathbf{z}\geq\mathbf{c}$.

Together (4) and the producer's first-order conditions imply:

Theorem 2 *Suppose that the directional input distance functions for firms* $(0,1)$ *are given by (3). For interior solutions,*

$$\frac{1}{2}\left(\frac{\delta^1\boldsymbol{\pi}^1\bullet\mathbf{p}^1}{\mathbf{w}^1\mathbf{g}}+\frac{\delta^0\boldsymbol{\pi}^0\bullet\mathbf{p}^0}{\mathbf{w}^0\mathbf{g}}\right)'\left(\mathbf{z}^1-\mathbf{z}^0\right)-\frac{1}{2}\left(\frac{\mathbf{w}^1}{\mathbf{w}^1\mathbf{g}}+\frac{\mathbf{w}^0}{\mathbf{w}^0\mathbf{g}}\right)'\left(\mathbf{x}^1-\mathbf{x}^0\right),$$

is exact for $L\left(\mathbf{x}^1,\mathbf{x}^0,\mathbf{z}^1,\mathbf{z}^0\right)$.

If there are objective probabilities or both producers have the same subjective evaluation of the states of Nature (rational expectations), then the result becomes even sharper.

Corollary 3 *Suppose that the directional input distance functions for firms* $(0,1)$ *are given by (3). If producers share common subjective probabilities* $(\boldsymbol{\pi})$ *and a common discount factor* (δ)*, then*

$$\left[\frac{\delta\boldsymbol{\pi}}{2}\bullet\left(\frac{\mathbf{p}^1}{\mathbf{w}^1\mathbf{g}}+\frac{\mathbf{p}^0}{\mathbf{w}^0\mathbf{g}}\right)\right]'\left(\mathbf{z}^1-\mathbf{z}^0\right)-\frac{1}{2}\left(\frac{\mathbf{w}^1}{\mathbf{w}^1\mathbf{g}}+\frac{\mathbf{w}^0}{\mathbf{w}^0\mathbf{g}}\right)'\left(\mathbf{x}^1-\mathbf{x}^0\right)$$

is exact for $L\left(\mathbf{x}^1,\mathbf{x}^0,\mathbf{z}^1,\mathbf{z}^0\right)$.

Finally, if both producers face the same decision environment and the same subjective evaluation of the states of Nature, our exactness result attains it sharpest and perhaps most intuitive focus:

Corollary 4 *Suppose that the directional input distance functions for firms* $(0,1)$ *are given by (3). If producers share common subjective probabilities* $(\boldsymbol{\pi})$ *a common discount factor* (δ)*, and a common decision environment* $(\mathbf{p}^1 = \mathbf{p}^0, \mathbf{w}^1 = \mathbf{w}^0)$*, then the difference in their present-value profit as normalized by the value of the reference input vector is exact for* $L(\mathbf{x}^1, \mathbf{x}^0, \mathbf{z}^1, \mathbf{z}^0)$.

The results are perhaps most easily grasped in reverse order. Corollary 4 informs us that, in a rational-expectations equilibrium, the cross-sectional differences between two risk-neutral firms present-value profit is a superlative indicator of their productivity differences. Put in more financial terminology, the difference in the firms' market values is a superlative indicator of their productivity. Thus, we obtain the intuitive proposition that more productive firms should be more valuable.

In a sense, this should be true almost by definition in the presence of arbitrage-free financial markets. However, the existing literature on the computation of economic productivity indexes (or on the pricing of financial assets) reveals little if any direct theoretical connection between a firm's market value and its relative productivity. This stands in stark contrast to the more popular economics and finance literatures where the commonsense explanation of market values (and hence stock prices in a stock market economy) routinely relies on references to greater productivity. Witness to this is the fact that one of the most popular (albeit *ex post)* rationalizations of the remarkable appreciation in share values in the late 1990s was an apparent productivity boom. Corollary 4 formalizes the link in a compact and hopefully intuitive fashion.

Corollary 3 covers the case where the units being compared share common subjective probabilities but face different market structures. This case seems particularly relevant to intertemporal productivity measurement. In that case, a superlative indicator of productivity differences is offered by the expected value of a Bennet-Bowley (quantity) indicator.[13] Chambers (2002) has shown that the Bennet-Bowley indicator is superlative for nonstochastic Luenberger productivity indicators. Corollary 3 and Theorem 2, thus, generalize those earlier results. Because the Bennet-Bowley indicator is also interpretable as the difference in

[13] The measure $\left[\left(\frac{\mathbf{p}^1}{\mathbf{w}^1\mathbf{g}} + \frac{\mathbf{p}^0}{\mathbf{w}^0\mathbf{g}}\right)\right]'(\mathbf{z}^1 - \mathbf{z}^0)$ can also be recognized as a normalized version of the Hicks' many-market consumer surplus measure (Chambers, 2001).

0 and $1's$ profit evaluated at the normalized prices $\left(\frac{\mathbf{p}^1}{\mathbf{w}^1\mathbf{g}}+\frac{\mathbf{p}^0}{\mathbf{w}^0\mathbf{g}}\right)$ and $\left(\frac{\mathbf{w}^1}{\mathbf{w}^1\mathbf{g}}+\frac{\mathbf{w}^0}{\mathbf{w}^0\mathbf{g}}\right)$, Corollary 3 has an interpretation parallel to Corollary 4 in those terms.

Theorem 2 addresses the case where firms have different subjective beliefs about Ω and face different market situations. There, $\delta\boldsymbol{\pi}_s$ represents the present value to the firm of one unit of consumption in state s of period 1. Thus, $\delta\boldsymbol{\pi}_s p_s$ represents the firm's present-value price of output sold in that state of Nature. Thus, the superlative productivity measure in Theorem 2 is the normalized difference in the firms' present-value profit as evaluated at the average of their respective present-value prices and input prices.

It is of interest to note that Theorem 2, under suitable reinterpretation, also represents a superlative productivity indicator for firms with general risk preferences. If each $\boldsymbol{\pi}^i$ in the theorem is interpreted as the respective firm's 'risk-neutral probability' (its endogenous marginal present-value for a unit of consumption in state s), the indicator is then exact for general risk preferences (Chambers, 2003). In either that case or in the present case where the $\boldsymbol{\pi}^i$ represent the firm's subjective probabilities, the empirical challenge is to elicit measures of these different probability distributions on Ω. Chambers (2003) explains how estimable productivity indicators can be derived in such circumstances under further assumptions on the firms' equilibrium behavior.[14]

4 An Empirical Application

From Corollary 4, the calculation of cross-sectional superlative productivity indicators is particularly simple when rational-expectations equilibria prevail. Simply compare the firms' market values deflated by the value of the reference input bundle. Our empirical application, however, more closely approximates Corollary 3. It involves calculating intertemporal stochastic productivity indicators for data similar to those that were used to derive Figure 1. These data are aggregate time series, in index form, for US agriculture for the period 1948 through 1999. There are time-series on a single aggregate output, four aggregate in-

[14] Here the presence of financial markets is crucial. Under the assumption that $\mathbf{c}-\mathbf{z}$ as chosen by the producer lies within the span of $\mathbf{A}$, these risk-neutral probabilities can be calculated as their projection onto $\mathbf{A}$.

puts (labor, nonland capital, materials, and land), and their respective prices.[15] The price indexes are constructed relative to a base year of 1996, and the associated quantity indexes are derived by dividing input or output values by the relevant price index. The empirical goal is to produce an intertemporal stochastic productivity indicator for US agriculture that is consistent with the theoretical development. In what follows, I presume the existence of a representative producer, with a stable probability measure for Ω, producing a single aggregate agricultural output using these four aggregate inputs.

Luenberger productivity indicators and Bennet-Bowley indicators are likely unfamiliar to many readers. Therefore, I first construct the nonstochastic Bennet-Bowley indicator for the data that is superlative for a nonstochastic Luenberger productivity indicator.[16] The nonstochastic Bennet-Bowley indicator for $(t+1,t)$ is[17]

$$\left[\frac{\delta}{2}\left(\frac{p^{t+1}}{\mathbf{w}^{t+1}\mathbf{g}}+\frac{p^{t}}{\mathbf{w}^{t}\mathbf{g}}\right)\right]\left(z^{t+1}-z^{t}\right)-\frac{1}{2}\left(\frac{\mathbf{w}^{t+1}}{\mathbf{w}^{t+1}\mathbf{g}}+\frac{\mathbf{w}^{t}}{\mathbf{w}^{t}\mathbf{g}}\right)'\left(\mathbf{x}^{t+1}-\mathbf{x}^{t}\right), \tag{5}$$

where (p^t, z^t) correspond, respectively, to price and quantity of the aggregate output at time t. Note, if prices and production are nonstochastic, (p^t, z^t) are scalars rather than state-contingent vectors.

This productivity indicator intends to measure, albeit from a different perspective, essentially the same phenomena as the logarithmic difference of total factor productivity presented pictorially in Figure 1. The two measures differ conceptually. The Luenberger nonstochastic indicator is a difference-based measure that relies on directional distance functions to represent the technology, while the logarithmic difference measure is a ratio-based productivity measure that relies on radial distance functions to represent the technology.

[15] These data were provided by V. Eldon Ball of ERS/USDA and were derived from the data available at www.ers.usda.gov/data/AgProductivity. They are available upon request.

[16] This indicator is derived theoretically in Chambers (1996, 2002).

[17] Because the data used in the analysis were developed in a static framework, there is no clear distinction between the time period in which inputs were chosen and committed and the period in which output and its price are realized. To enhance comparability with existing productivity measures, I use the following convention: $\mathbf{w}^t$ and $\mathbf{x}^t$ are the input prices and inputs chosen in the planning period (0) and z^t is the output realized in period (1) .One might argue, with legitimate force, that a more proper choice of $\mathbf{w}^t$ would actually be $\mathbf{w}^{t-1}$. However, this would involve a different valuation of utilized inputs than in current productivity measurement.

The reference input vector, $\mathbf{g}$, was chosen to be the sample mean for the data set. The period-to-period productivity indicators, therefore, intend to approximate the inward or outward shifts of the frontier of the technology in units of the mean input vector. The discount factor was arbitarily set to 1 to enhance comparability with the measures reported in Figure 1.

Figure 2 depicts the resulting time series of the calculated Bennet-Bowley indicator and the logarithmic differences in total factor productivity presented in Figure 1 on the same axes. These time series are virtually indistinguishable. Thus, while the method of measurement differs slightly, both measures tell the same tale. From the 1970s on, measured growth in US agricultural productivity oscillated widely and experienced a number of sharp declines and equally sharp upswings. In what follows, I restrict attention to Bennet-Bowley indicators.

Recognizing that the technology and the decision environment is uncertain, Corollary 3 applies. The task for $(t+1, t)$ is to measure

$$\left[\frac{\delta \boldsymbol{\pi}}{2} \bullet \left(\frac{\mathbf{p}^{t+1}}{\mathbf{w}^{t+1}\mathbf{g}} + \frac{\mathbf{p}^{t}}{\mathbf{w}^{t}\mathbf{g}}\right)\right]' \left(\mathbf{z}^{t+1} - \mathbf{z}^{t}\right) - \frac{1}{2}\left(\frac{\mathbf{w}^{t+1}}{\mathbf{w}^{t+1}\mathbf{g}} + \frac{\mathbf{w}^{t}}{\mathbf{w}^{t}\mathbf{g}}\right)' \left(\mathbf{x}^{t+1} - \mathbf{x}^{t}\right).$$

Rewriting the leftmost term in an explicit expectation notation gives

$$\left[\frac{\delta \boldsymbol{\pi}}{2} \bullet \left(\frac{\mathbf{p}^{t+1}}{\mathbf{w}^{t+1}\mathbf{g}} + \frac{\mathbf{p}^{t}}{\mathbf{w}^{t}\mathbf{g}}\right)\right]' \left(\mathbf{z}^{t+1} - \mathbf{z}^{t}\right) = \frac{\delta}{2} E\left[\left(\frac{\tilde{p}^{t+1}}{\mathbf{w}^{t+1}\mathbf{g}} + \frac{\tilde{p}^{t}}{\mathbf{w}^{t}\mathbf{g}}\right)\left(\tilde{z}^{t+1} - \tilde{z}^{t}\right)\right]$$

where, for example, $\tilde{z}^t$ represents the stochastic output at time t and E represents expectation. Thus, in a random-variable notation, the superlative indicator is

$$\frac{\delta}{2} E\left[\left(\frac{\tilde{p}^{t+1}}{\mathbf{w}^{t+1}\mathbf{g}} + \frac{\tilde{p}^{t}}{\mathbf{w}^{t}\mathbf{g}}\right)\left(\tilde{z}^{t+1} - \tilde{z}^{t}\right)\right] - \frac{1}{2}\left(\frac{\mathbf{w}^{t+1}}{\mathbf{w}^{t+1}\mathbf{g}} + \frac{\mathbf{w}^{t}}{\mathbf{w}^{t}\mathbf{g}}\right)' \left(\mathbf{x}^{t+1} - \mathbf{x}^{t}\right). \tag{6}$$

The rightmost term in (6), which measures the contribution of input change to the productivity indicator, remains calculable just as before. Without further assumptions, however, the leftmost term is not computable. There are a number of approaches that one might take in measuring the leftmost term.

A structural approach would solve the firm's first-order conditions for $(\mathbf{x}, \mathbf{z})$ period by period and then construct the resulting expectation using the solutions. This approach is particularly onerous for a number of reasons. First, it requires a specification of the directional distance function to be solved. One might think to simply choose the quadratic form above, but that form needs to be augmented by a stationarity assumption specifying how the technology evolves with time. Specifically, the dependence of the constant and linear terms in (3) on time would need to be specified. This requires a parameterization of the effect of technical change on the technology. Explicit distributional assumptions also need to be made for $\mathbf{p}^t$ and $\mathbf{p}^{t+1}$ to permit a closed form solution of the problem. Even within the quadratic specification, the ultimate solution can be highly nonlinear in the absence of simplifying assumptions. To the extent that any of these assumptions are wrong, the approach would lack robustness. These points are not meant to suggest that these alternatives should not ultimately be pursued to their fullest. Quite the opposite is the case. However, they do suggest that a less ambitious and more straitghtforward approach that is aimed solely at estimating the stochastic Bennet-Bowley indicator may prove valuable.

The approach pursued in this study is, perhaps, the most obvious and, therefore, perhaps the most simple. The recognition that empirical implementation of the stochastic Bennet-Bowley indicator requires point-by-point knowledge of the expectation in (6) suggests that one might profitably postulate a regression relationship of the general form

$$E\left[\left(\frac{\tilde{p}^{t+1}}{\mathbf{w}^{t+1}\mathbf{g}}+\frac{\tilde{p}^{t}}{\mathbf{w}^{t}\mathbf{g}}\right)\left(\tilde{z}^{t+1}-\tilde{z}^{t}\right)\right]=\mathbf{V}^{t+1}\boldsymbol{\beta}, \tag{7}$$

where $\mathbf{V}^{t+1}$ is a $(1\times K)$ vector of explanatory variables, which can be taken as predetermined at $t+1$, and $\boldsymbol{\beta}$ is a $(K\times 1)$ vector of parameters. Given (7), observations on the explanatory variables $\mathbf{V}^{t+1}$ and on the observed value of the Bennet-Bowley revenue indicator,

$$\left[\frac{1}{2}\left(\frac{p^{t+1}}{\mathbf{w}^{t+1}\mathbf{g}}+\frac{p^{t}}{\mathbf{w}^{t}\mathbf{g}}\right)\right]\left(z^{t+1}-z^{t}\right), \tag{8}$$

can be used to estimate $\boldsymbol{\beta}$ to obtain, in turn, an estimate of the expectation as $\mathbf{V}^{t+1}\hat{\boldsymbol{\beta}}$.

The most obvious candidates for the explanatory variables in this regression relationship are the inputs that condition the empirical distribution of the stochastic output. Because they are sector-level inputs, the price distribution may also depend upon these input levels. In the analysis, therefore, $\mathbf{V}^{t+1}$ was taken to be the vector input differences, $\mathbf{x}^{t+1}-\mathbf{x}^{t}$, as well as a constant term.[18]

[18]Results in terms of input levels, as opposed to differences, were essentially similar to those reported below. Similarly, results including input differences as well as subvectors of the instrumental variables below were essentially similar to what is reported below.

There are some obvious potential problems with using the observed inputs to explain the observed value of the Bennet-Bowley indicator. Most importantly, in a state-space framework, the inputs and the state-contingent outputs are simultaneously chosen by the producer. Hence, there exists the possibility that the inputs will be correlated with any error structure that might be associated with (7).

Two separate estimation strategies were, therefore, pursued. The first ignored the potential simultaneity of the inputs and regressed (8) on $\mathbf{V}^{t+1}$ using ordinary least squares (OLS). The OLS parameter estimates were then used to generate estimates of the point-by-point expected values for 1949 to 1999. The second approach used a two-stage instrumental variables (IV) procedure. In the first stage, the input differences were regressed upon time-series observations on a set of predetermined variables from the macro economy (real gross domestic product, the consumer price index, the natural logarithm of US population, and a time trend) to purge them of possible simultaneity with the regression error, and in the second stage the purged predictions of the input differences were used to estimate the parameter vector $\boldsymbol{\beta}$.[19] The estimated parameter vector was then used along with $\mathbf{V}^{t+1}$ to obtain estimates of the expectation.

Figure 3 depicts the point estimates of the stochastic Bennet-Bowley indicator derived from the single-stage OLS regressions and the nonstochastic Bennet-Bowley indicator (5) on the same axes. Figure 4 depicts the estimates of the stochastic productivity indicator derived from the two-stage IV and the nonstochastic Bennet-Bowley productivity indicator on the same axes. Both figures lead to a similar conclusion. The point estimates of the stochastic productivity indicator suggest much less variability in productivity change than the nonstochastic productivity indicator. We focus our remaining attention on the potentially consistent estimates derived from the IV procedure.

The stochastic productivity indicator in Figure 4, being almost continually positive, suggests virtually continuous productivity growth throughout the sample period. In contrast,

[19] One might think to use observed input prices as natural instruments. If these data had been obtained directly from market observations, given the behavioral assumptions of this model, this would seem the most preferable approach. However, as noted above, the price data are aggregate price indexes which are dependent upon the quantities that are summarized by the index, and the quantity indexes are derived using the price indexes to deflate input expenditures.

the nonstochastic productivity indicator suggests overall growth punctuated by episodes of sharply negative productivity growth. One crude indicator of the difference between these measures is given by simply counting the number of periods in which the two indicators report a decline in agricultural productivity (negative growth). Over the period from 1970 to 1999, the nonstochastic productivity indicator reports 10 such declines, with four of these exceeding .05 of average input use and one approaching .15 of the average input vector. (Recall from Figure 2 that the official USDA measures tell the same story.) The stochastic productivity indicator, on the other hand, suggest that productivity only fell in one of these periods, and then by well less than .05 units of the average input vector. Particularly remarkable is the fact that the stochastic productivity indicator reports productivity growth in all the periods after 1970 in which the nonstochastic indicator reports steep declines (those exceeding .05 of the average input vector) in agricultural productivity (negative growth).

Figure 5 documents this greater stability in a slightly different way. There I report a crude intertemporal productivity indicator that permits a direct comparison of the initial sample period (1948) with the last sample period (1999) and all intervening periods. This cumulative productivity indicator is constructed as follows. Base period productivity is set arbitrarily to zero and then successive periods are compared to the base period by adding the intervening period-by-period indicators of productivity change. Thus, for example, the estimate of the cumulative productivity indicator for 1950 is given by the sum of the estimated productivity indicator comparing 1948 and 1949 and the estimated productivity indicator for 1949 and 1950, and the cumulative productivity indicator for 1999 by the sum of all the period-by-period productivity indicators reported in Figure 4.[20]

Figure 5 compares the resulting cumulative productivity indicators for the estimated stochastic productivity indicator and the nonstochastic Bennet-Bowley indicator. Because the regression analysis includes a constant term, by construction, both of the cumulative productivity indicators start and end at the same place. However, Figure 5 suggests very

[20] These measures are crude because no attempt was made to ensure that the Luenberger productivity indicator satisfied the transitivity required to permit such simple adding up. Chambers (1998) and Fox (2003) contain detailed discussions and results on transitive difference-based productivity indicators. It is worthwhile to note that total factor productivity measures are derived in an analogous fashion. A base period is set to 1. Percentage changes are then compounded multiplicatively.

different routes to the final destination. The cumulative stochastic productivity indicator suggests that agricultural productivity grew relatively continuously over the sample period with relatively few absolute declines in estimated productivity. Moreover, when declines did occur, they were much more damped than the nonstochastic productivity indicator would suggest.

Visual inspection of Figures 1 to 4 reveals another interesting observation. Every instance of a measured productivity decline (negative growth) by the nonstochastic Bennet-Bowley indicator or the differenced Malmquist measure (5) was followed immediately by an instance of measured productivity growth. And, in more instances than not, the steeper the measured productivity decline, the sharper the following measured productivity growth. This strikes me as intuitive—provided that the nonstochastic productivity indicator is conflating true productivity changes with the effects of varying states of Nature.

Consider the following thought experiment. For a given level of inputs, if Nature makes a very 'bad' draw for the producer from Ω, the likely observed effect is a very low output. If input use remains stable for the following period, *the technology remains unchanged*, and Nature follows such a 'bad' draw with a normal or even a moderately bad draw from Ω, the likely observed effect is a sharp increase in output. The result is a sharp upswing in measured productivity. However, it is apparent that the measured productivity increase captures not a sustained change in productive ability, but the randomness of production. Similar logic applies if Nature suddenly makes a very 'good' draw from Ω. Thus, to the extent that Nature is making fairly random draws from Ω, measured productivity growth could inherently exhibit a 'teeter-totter' effect.

This phenomenon is perhaps more visually accessible in Figure 6 which portrays the period-by-period differences between the two measures depicted in Figure 4. Because the input component of the two productivity measures is the same, this difference corresponds to the estimated residuals from the regression relationship.[21] In a sense, therefore, Figure

[21] If the producer has general risk preferences the difference between the nonstochastic indicator and the stochastic indicator will not correspond to the residuals of a regression relationship (Chambers, 2003). Hence, in that instance, the differences are perhaps more accurately thought of as measuring the role that stochastic factors play in determining productivity.

6 depicts estimates of the 'error' associated with using a nonstochastic approach when the technology and decision environment are truly stochastic. Given the crudeness of the empirical methodology, that interpretation is manifestly overambitious and won't withstand close scrutiny. However, it is reasonable to interpret this figure as measuring, albeit imperfectly, the role that stochastic factors, which are not properly accounted for, can play in nonstochastic productivity measurement.

To investigate this serial adjustment, I fit via OLS a simple first-order autoregressive model to the data depicted in Figure 6. Denote the observed values in that figure by u_t. The resulting estimated model is

$$u_t = -\underset{(.1198)}{.5223} u_{t-1},$$

where the terms in parentheses is the estimated standard error.[22] There does appear to be significant serial correlation in the estimated error structure for (7). This crudely confirms our visual interpretation of the data. When nonstochastic productivity underestimates the stochastic productivity measure in period t, *ceteris paribus,* it will tend to overestimate it in period $t+1$ by approximately 52% of the previous underestimate, underestimate it in period $t+2$ by about 27%, etc. until the effect of the innovation finally disappears.

These results suggest that it may be possible to improve the asymptotic efficiency of the estimation procedure by accounting for this autoregressive error structure in the IV estimation of (7). Figure 7 compares stochastic productivity estimates obtained by incorporating this estimated autoregressive error structure into a single-step feasible generalized least squares IV estimation procedure with the stochastic Bennet-Bowley indicator reported in Figure 4. The results are quite similar to those described earlier. The main difference is that the autocorrelation corrected estimates report two periods of post 1970 negative productivity growth as opposed to one.

Figures 3 through 7 suggest a very different story about productivity change in US agri-

[22] Estimation of the analogous autoregressive structure, but with drift, for the nonstochastic Luenberger productivity indicator, denoted by l_t, yielded

$$l_t = \underset{(.0064)}{.0265} - \underset{(.1203)}{.5152}.$$

culture than Figures 1 or 2. Figures 1 and 2 portray an industry whose productivity frontiers are shifting back and forth in a very rapid and sometimes violent manner. Figures 3 through 7 indicate that US agriculture is an industry that has exhibited sustained and almost continuous productivity growth throughout the postwar era. Which story is correct? As usual when there are such divergent results, the likely answer is probably neither. There are many obvious caveats to the brief empirical analysis presented here. In particular, the stochastic productivity analysis here has been conducted at a very high level of aggregation using a theory that is explicitly predicated upon individual behavior. Even though this practice is truly ubiquitous in aggregate productivity analysis (or in other fields, such as finance, for that matter), it does imply that the results cannot be judged as definitive. Indeed, the empirical results are not intended as a definitive description of postwar productivity changes in US agriculture. The smoothing procedure, after all, is very rudimentary. What they hopefully do indicate, however, is that once productivity measurement takes into account the inherently stochastic world producers face, existing productivity measures need to be reassessed. The same data can seemingly support a compellingly different interpretation of productivity change in the postwar era than the one suggested by Figure 1.

5 Conclusion

Stochastic productivity indicators have been defined, and superlative measures of these indicators have been derived for risk-neutral producers. It has been shown that in a rational-expectations equilibrium, differences in the market values of firms are superlative indicators of cross-sectional productivity differences. An illustrative regression-based method for estimating the intertemporal superlative productivity indicators has been discussed and applied to data for postwar US agriculture. Comparison with existing nonstochastic productivity indicators suggests that properly accounting for the stochastic environment in which firms operate could have important empirical implications for measuring productivity growth.

The current discussion has been restricted to the specification and approximation of stochastic productivity indicators and indices. However, because of the intimate connection between these indicators and input and output quantity indicators and indices (and by extension implicit price indicators), it is obvious that our analysis applies equally well to such concepts. The extension of our results to such measures is left to the interested reader.

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Figure 1: Logarithmic Changes in Agricultural Total Factor Productivity

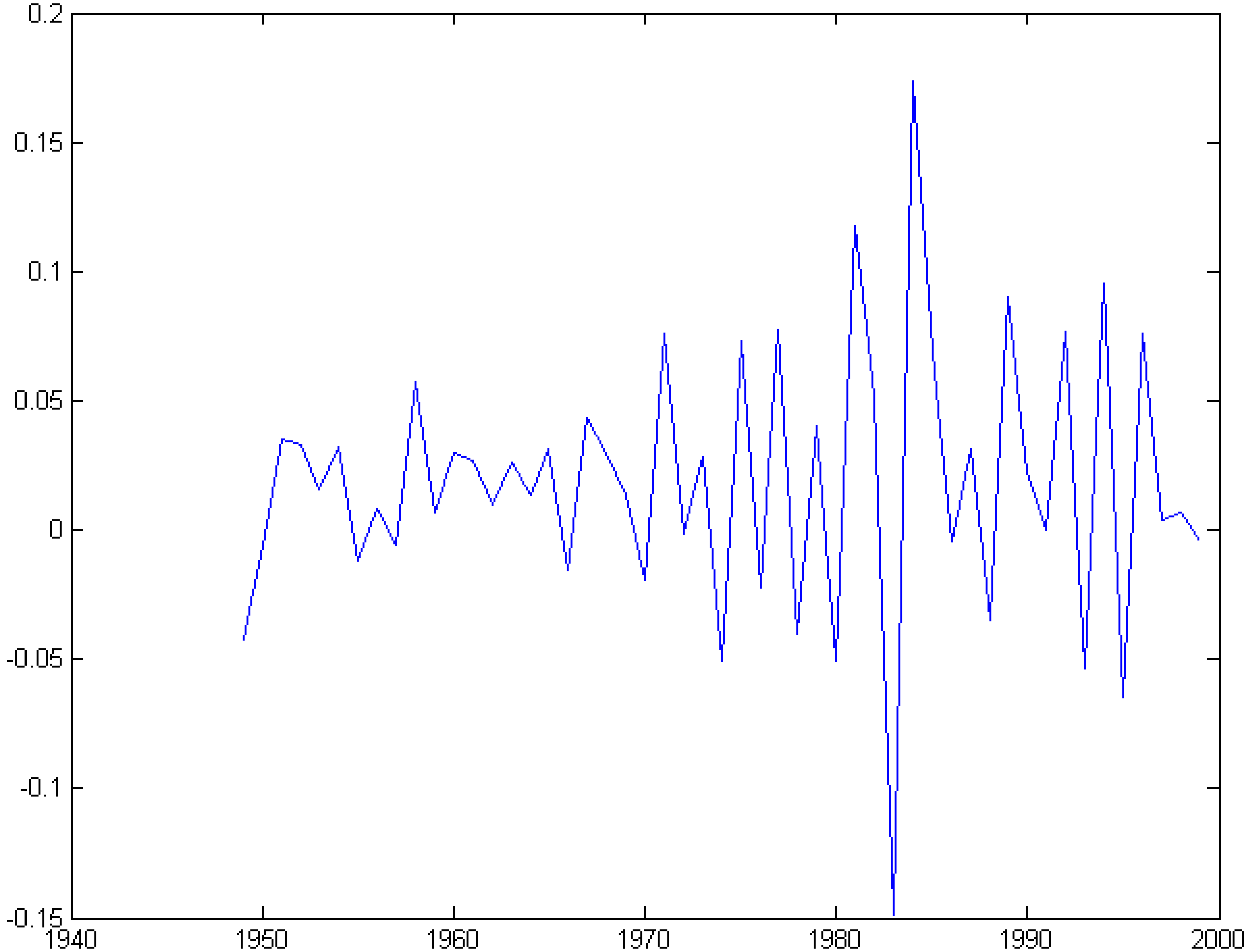
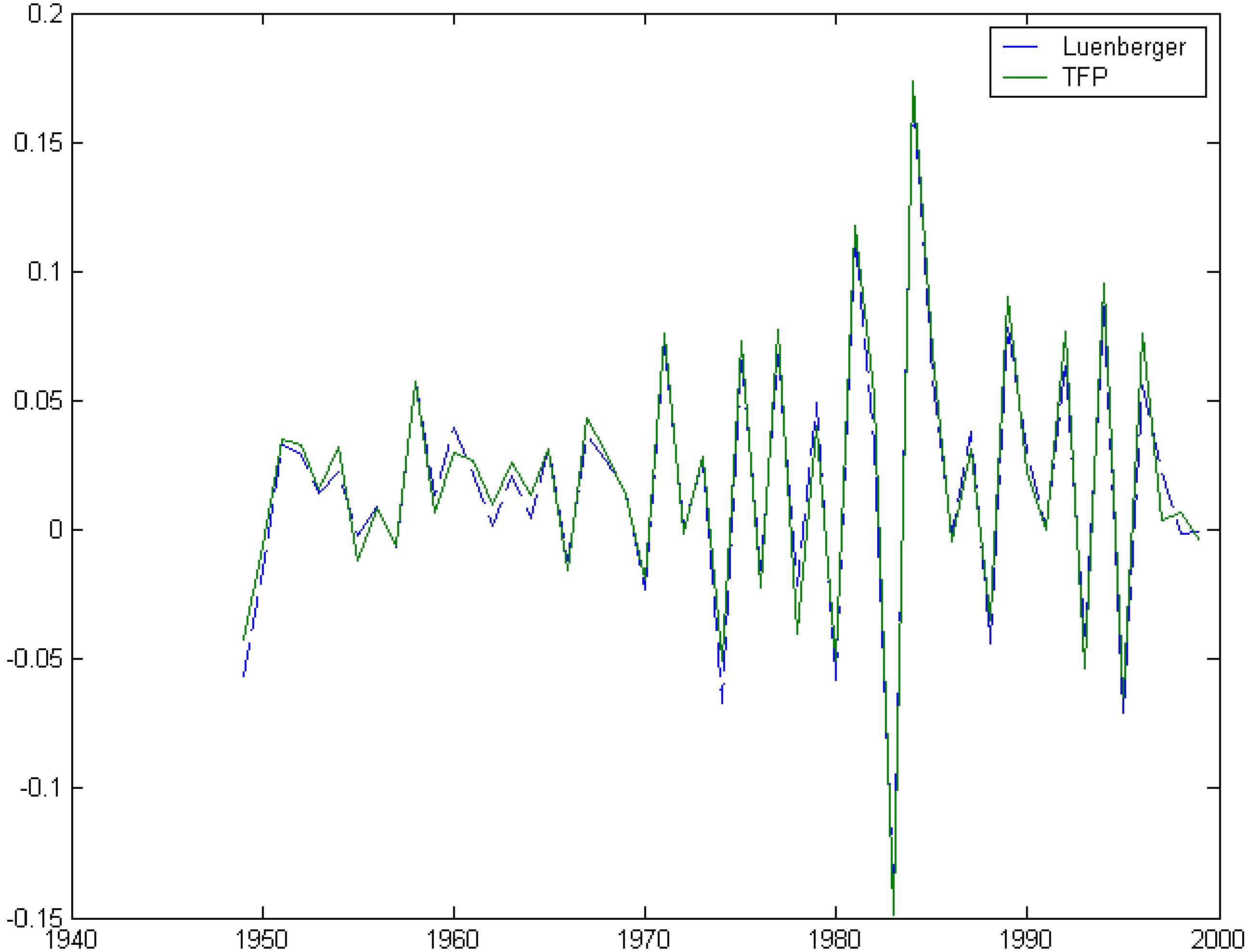


Figure 2:Nonstochastic Luenberger Productivity Indicator and TFP



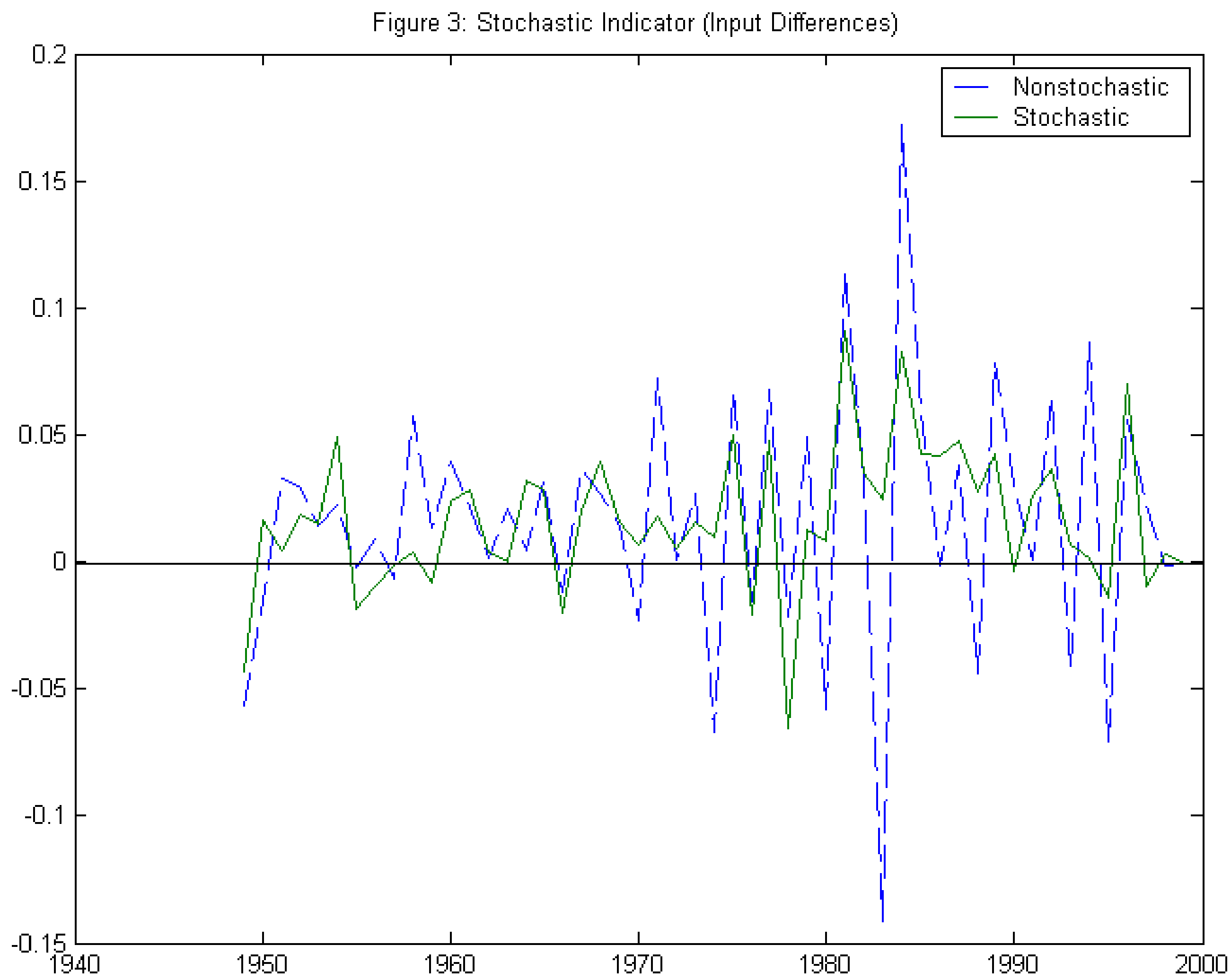


Figure 3: Stochastic Indicator (Input Differences)
Nonstochastic
Stochastic
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
1940
1950
1960
1970
1980
1990
2000

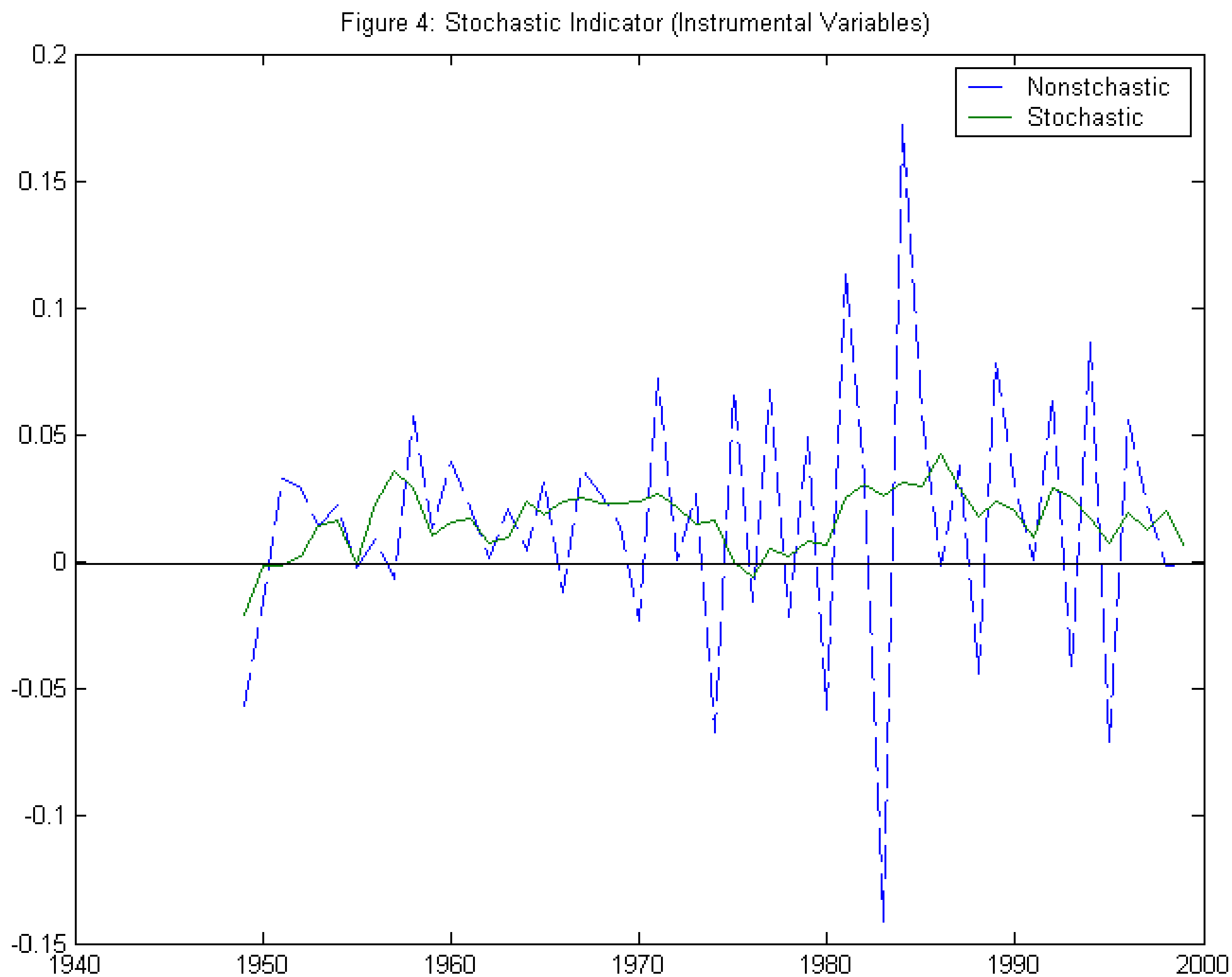
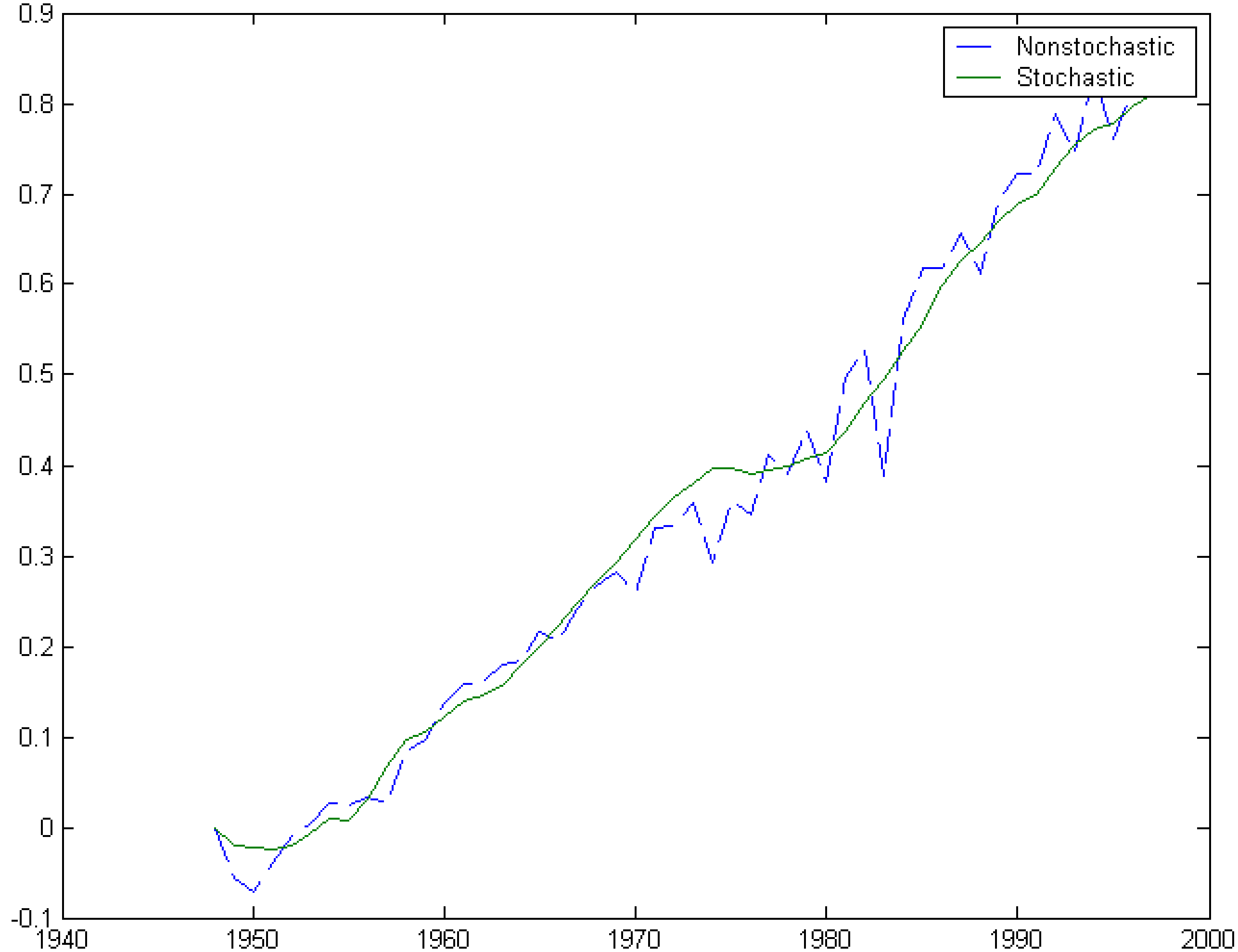


Figure 4: Stochastic Indicator (Instrumental Variables)
Nonstchastic
Stochastic
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
1940
1950
1960
1970
1980
1990
2000

Figure 5: Cumulative Productivity Measures



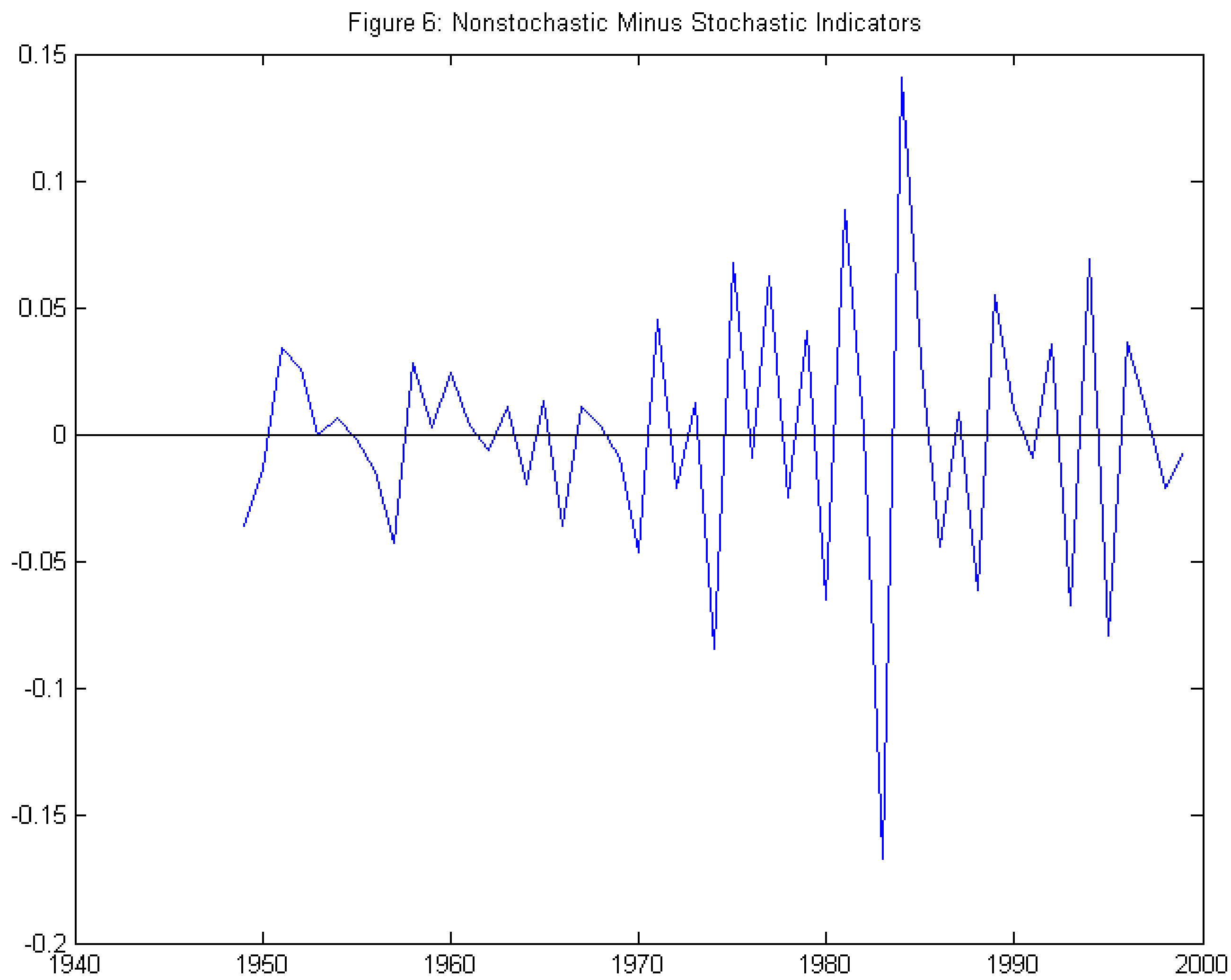


Figure 6: Nonstochastic Minus Stochastic Indicators
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
1940
1950
1960
1970
1980
1990
2000

Figure 7: Autocorrelation Corrected Stochastic Productivity Indicator