Direct and Chained Indices: A Review of Two Paradigms

Bert M. Balk[*]
Erasmus Research Institute of Management
Erasmus University Rotterdam
and
Statistics Netherlands
P. O. Box 4000, 2270 JM Voorburg, Netherlands
bblk@cbs.nl

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1. Introduction

A recurrent theme when measuring aggregate price and quantity change between more than two periods is the choice between the computation of direct or chained index numbers. Suppose we consider periods 0, 1, 2, …,T and wish to measure change relative to the base period 0. A direct index number comparing period t ($t = 1,\ldots,T$) to period 0 is the result of inserting period t and period 0 data in a bilateral index formula. A chained index number comparing period t to period 0 results when we insert period 1 and period 0 data, period 2 and period 1 data, …., period t and period t-1 data respectively in a bilateral index formula and multiply the outcomes with each other.

A basic motivation for the method of chaining appears to be the reduction of so-called index number spread. As the *CPI Manual* (2004) states

> "The main advantage of the chain system is that under normal conditions, chaining will reduce the spread between the Paasche and Laspeyres indices." (par. 15.83)
> "Basically, chaining is advisable if the prices and quantities pertaining to adjacent periods are *more similar* than the prices and quantities of more distant periods, since this strategy will lead to a narrowing of the spread between the Paasche and Laspeyres indices at each link." (par. 15.85)

The detailed numerical example discussed in chapter 19 confirms this viewpoint, as the following quotations make clear:

> " … if the underlying price and quantity data are subject to reasonably smooth trends over time, then the use of chain indices will narrow considerably the dispersion in the asymmetrically weighted indices." (par. 19.16)
> " … the combined effect of using both the chain principle as well as symmetrically weighted indices is to dramatically reduce the spread between all indices constructed using these two principles." (par. 19.21)

The overall impression one gets is that chained index numbers are somehow closer to the truth than direct index numbers. But is this impression warranted?

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The technique of chaining index numbers was introduced by Lehr (1885) and Marshall (1887), primarily as a means to overcome the problem of comparing distant periods when there are many disappearing and newly appearing commodities through time. Statistical agencies, however, have always been reluctant to officially use chained index numbers. During the last two decades the situation has started changing.
This change was not caused by some convincing theoretic demonstration of the 'verisimilitudiness' of the method of chaining, but rather due to what could be called climatic factors. Under the influence of a small number of researchers, some important agencies in the field of economic measurement changed opinion.

Both the recommendation of chaining and the replacement of Laspeyres and Paasche (asymmetrically weighted) by Fisher (symmetrically weighted) indices has become the focal point of the criticism voiced by Peter von der Lippe[1]. Von der Lippe appears to divide the world between "chainers" and "non-chainers". The discussion often exhibits features reminiscent of the many Mediaeval, scholastic disputes. In the framework of this paper it is not feasible, but happily also not necessary, to review all the arguments put forward by Von der Lippe and others. I will concentrate on the core issues.

The plan of this paper is as follows. Section 2 summarizes the, what I call, traditional point of view which is based on the use of direct Laspeyres and Paasche indices. Section 3 likewise summarizes the modern point of view, based on the use of chained Fisher indices. In section 4 both views are compared, leading to the conclusion that, mathematically seen, a unification of the two is impossible. When, as it seems to be the case, the strategy of chaining is by and large motivated by practical reasons, an important question remains unanswered: what precisely does a chained price or quantity index measure? In sections 5 and 6 I am searching for an answer, in section 5 by using micro-economic theory, and in section 6 by using Divisia index theory. Section 7 concludes.

2. The traditional point of view

I consider an economic aggregate, consisting of a number of transaction categories which I will call 'commodities'. For the time being, I will assume that these commodities do not change through time. Each of these commodities has an (average) price p_n^t and a corresponding quantity q_n^t, where $n = 1,\ldots,N$ denotes a commodity, and t is an accounting period (thought of as being a year). The (transaction) value of commodity n in period t is then $v_n^t \equiv p_n^t q_n^t$, and the value of the entire aggregate is $V^t \equiv \sum_{n=1}^{N} p_n^t q_n^t$. It is efficient to use from hereon simple vector notation. Hence, $p^t \cdot q^{t'} \equiv \sum_{n=1}^{N} p_n^t q_n^{t'}$, where t and t' denote two, not necessarily different, time periods.

Consider now the development of this aggregate through a number of consecutive periods, say $t = 0, 1, 2, \ldots, T$. One then obtains the sequence of nominal values

$$p^0 \cdot q^0,\ p^1 \cdot q^1,\ p^2 \cdot q^2, \ldots,\ p^T \cdot q^T . \qquad (1)$$

[1] See Von der Lippe (2000), (2001a), Reich (2000), Von der Lippe (2001b), and Rainer (2002). The discussion appears to be by and large limited to the readership of the *Allgemeines Statistisches Archiv.*

It is clear that the nominal value development is caused by price and quantity changes. The problem is to disentangle the two components in order to get a picture of the 'real' development.

The traditional solution[2] consists in transforming the sequence of nominal values into a sequence of values-at-constant-prices. If one employs the period 0 prices as constant prices, the solution becomes that of computation of the sequence

$$p^0 \cdot q^0,\ p^0 \cdot q^1,\ p^0 \cdot q^2, \ldots,\ p^0 \cdot q^T. \qquad (2)$$

In practice, the computation is carried out elementwise in two ways. One way is to multiply (inflate) each commodity's nominal period 0 value by its quantity change,

$$p_n^0 q_n^0 (q_n^t / q_n^0) = p_n^0 q_n^t \quad (t = 1,\ldots,T). \qquad (3)$$

The other way is to divide (deflate) each commodity's period t value by its price change,

$$p_n^t q_n^t / (p_n^t / p_n^0) = p_n^0 q_n^t \quad (t = 1,\ldots,T). \qquad (4)$$

The adding-up of $p_n^0 q_n^t$ for $n = 1,\ldots,N$ delivers $p^0 \cdot q^t$. With hindsight, the sequence (2) can be considered as having been obtained by deflating the sequence of nominal values (1) after the first period by Paasche price index numbers, or by inflating the first period nominal value by Laspeyres quantity index numbers,

$$p^0 \cdot q^t = p^t \cdot q^t / P_P(t,0) = p^0 \cdot q^0 Q_L(t,0) \quad (t = 1,\ldots,T). \qquad (5)$$

Recall that the Laspeyres price index is defined by $P_L(t,t') \equiv p^t \cdot q^{t'} / p^{t'} \cdot q^{t'}$, the Paasche price index by $P_P(t,t') \equiv p^t \cdot q^t / p^{t'} \cdot q^t$, the Laspeyres quantity index by $Q_L(t,t') \equiv p^{t'} \cdot q^t / p^{t'} \cdot q^{t'}$, and the Paasche quantity index by $Q_P(t,t') \equiv p^t \cdot q^t / p^t \cdot q^{t'}$.

The aggregate quantity change between any two periods can be computed simply by taking the ratio of the two values from the sequence (2). Usually one is interested in the change between two adjacent periods t-1 and t. In this case the quantity change is given by

$$p^0 \cdot q^t / p^0 \cdot q^{t-1} = Q_{Lo}(t,t-1;0) \quad (t = 1,\ldots,T). \qquad (6)$$

This formula is an instance of what in the literature is known as a Lowe quantity index[3]. Its interpretation is straightforward: the numerator contains the period t quantities evaluated at their base period prices, and the denominator contains the period t-1 quantities evaluated at the same prices.

The whole construction (1), (2), (6) has the virtue of simplicity. The feast becomes a little bit disturbed, however, when one takes a closer look at the price component that corresponds to the quantity component (6). This price component is obtained by dividing the value change by the quantity change,

[2] I associate this view with *SNA 1968*, the relevant paragraph being 4.46.
[3] Some are used to call this a 'modified Laspeyres quantity index'.

$$\frac{p^t \cdot q^t / p^{t-1} \cdot q^{t-1}}{p^0 \cdot q^t / p^0 \cdot q^{t-1}} \quad (t = 1,\ldots,T). \tag{7}$$

This formula not only is less simple than (6), but also has an important disadvantage. If between periods t-1 and t all prices change by the same factor, that is, if $p_n^t = \lambda p_n^{t-1}$ (n = 1,…,N) for a certain $\lambda > 0$, then formula (7) in general will exhibit an outcome different from λ. Hence, formula (7) is not a genuine price index.

In practice one also has to face all the difficulties connected with the fact that our assumption of (an) unchanging (set of) commodities is not valid. First, in the course of time new commodities enter the aggregate. The problem becomes clear by looking at formulas (3) and (4). For any new commodity, its base period value as well as quantity equals zero; hence, formula (3) cannot be used. Although its period t value as well as price is known, its base period price does not exist; hence, formula (4) cannot be used, too. Of course, for commodities that in the course of time have disappeared from the aggregate an analogous problem holds. Second, even when there are no (dis-) appearing commodities, one usually has to cope with the problem of quality change. Quality change of commodity n occurs when its period t price cannot immediately be compared to its base period price; or, equivalently, when its period t quantity cannot immediately be compared to its base period quantity. Dependent on the calculation method chosen – according to formula (3) or (4) – the quantity or price change must somehow be adjusted for the quality change that has occurred.

The important point is that in all these cases one has to make imputations or estimates, and that this becomes more difficult and more dubious the longer the time span between base period and period t becomes. In addition, with the lapse of time it becomes less and less meaningful to aggregate recent quantities with prices from a past period. Therefore, every five or ten years one migrates to a new set of constant prices, which causes structural breaks in the time series of values-at-constant-prices.

3. The modern point of view

The modern view originates at the fact that one's primary interest lies in measuring the real change between two adjacent periods. Put otherwise, the primary problem is to decompose the value change

$$p^t \cdot q^t / p^{t-1} \cdot q^{t-1} \quad (t = 1,\ldots,T) \tag{8}$$

in a price and a quantity component. There are various ways to do this. One frequently decomposes the value change into a Paasche price index and a Laspeyres quantity index

$$\frac{p^t \cdot q^t}{p^{t-1} \cdot q^{t-1}} = P_P(t,t-1)Q_L(t,t-1) \quad (t = 1,\ldots,T). \tag{9}$$

The axiomatic approach, however, leads to the recommendation[4] to use Fisher price and quantity indices; hence, to decompose the value change as

$$\frac{p^t \cdot q^t}{p^{t-1} \cdot q^{t-1}} = \left[\frac{p^t \cdot q^{t-1}}{p^{t-1} \cdot q^{t-1}} \frac{p^t \cdot q^t}{p^{t-1} \cdot q^t} \right]^{1/2} \left[\frac{p^{t-1} \cdot q^t}{p^{t-1} \cdot q^{t-1}} \frac{p^t \cdot q^t}{p^t \cdot q^{t-1}} \right]^{1/2}$$

$$\equiv P_F(t,t-1) Q_F(t,t-1) \quad (t = 1,\ldots,T). \tag{10}$$

The first part at the right hand side of the equality sign is the price index, and the second part is the quantity index. The peculiar feature is that both indices have the same functional form: by interchanging prices and quantities the indices turn into each other.

It may be clear that new commodities, disappearing commodities, and quality changes also cause problems in the computation of the components of (10). But, since the time span between periods t-1 and t is relatively small – usually a year –, the extent of the problems that must be solved is smaller than in the case discussed in the previous section: there are less new and disappearing commodities, and less (large) quality changes to account for when comparing two adjacent periods than two periods far apart.

Not so well known, but extremely useful is the fact that the Fisher quantity index can be written in a form comparable to formula (6). This result, for the first time discovered by Jan van IJzeren (1952), reads

$$Q_F(t,t-1) = \frac{\frac{1}{2}(p^{t-1} + p^t / P_F(t,t-1)) \cdot q^t}{\frac{1}{2}(p^{t-1} + p^t / P_F(t,t-1)) \cdot q^{t-1}} \quad (t = 1,\ldots,T). \tag{11}$$

The numerator contains the period t quantities valued at the average, deflated prices of periods t-1 and t, whereas the denominator contains the period t-1 quantities valued at the same prices. The reasoning behind the deflation is not too difficult to see: if one would not deflate, then the period with the highest price level would determine the valuation of the quantities. This formula enables one to view the aggregate quantity change, $Q_F(t,t-1)$, as a weighted arithmetic average of individual quantity changes, q_n^t / q_n^{t-1} ($n = 1,\ldots,N$).[5] This makes clear to what extent the various commodities contribute to the aggregate quantity change.

Does there exist in this approach an analogue to the sequence of values-at-constant-prices (2)? The answer appears to be: yes. Based on expression (5), the analogue to (2) is given by the sequence of real values

$$p^t \cdot q^t / P(t,0) = p^0 \cdot q^0 Q(t,0) \quad (t = 1,\ldots,T), \tag{12}$$

[4] A summary of the underlying literature can be found in Diewert (1996).

[5] See Balk (2004) for alternatives. Formula (11) is since 1999 in use by the U. S. Bureau of Economic Analysis; see Ehemann *et al.* (2002). Of course, a similar formula holds for the Fisher price index. Notice that each component of the vector $p^{t-1} + p^t / P_F(t,t-1)$ depends on all prices and all quantities.

where $P(t,t')$ is some price index and $Q(t,t')$ is some quantity index. Notice that (12) expresses in a slightly different way what in the axiomatic approach is called the Product Test.

Now *SNA 1993* recommends either to start at the left hand side of (12) and to deflate nominal values by chained Fisher price index numbers, *i.e.* to replace $P(t,0)$ by

$$P_F^c(t,0) \equiv \prod_{\tau=1}^{t} P_F(\tau,\tau-1) \quad (t = 1,\ldots,T), \tag{13}$$

or to start at the right hand side of (12) and to inflate the nominal base period value by chained Fisher quantity index numbers, *i.e.* to replace $Q(t,0)$ by

$$Q_F^c(t,0) \equiv \prod_{\tau=1}^{t} Q_F(\tau,\tau-1) \quad (t = 1,\ldots,T). \tag{14}$$

The real values one thus obtains correspond to what in the U.S.A. has come to be called 'chained dollars'[6]. The use of chained Fisher price index numbers (13) is consistent with (10); for, dividing the real values of two adjacent periods into each other, one obtains

$$\frac{p^t \cdot q^t / P_F^c(t,0)}{p^{t-1} \cdot q^{t-1} / P_F^c(t-1,0)} = \frac{p^t \cdot q^t / p^{t-1} \cdot q^{t-1}}{P_F(t,t-1)} = Q_F(t,t-1) \quad (t = 1,\ldots,T), \tag{15}$$

that is, the quantity change that has occurred between the two periods. The same holds for the use of chained Fisher quantity index numbers (14).

An alternative was proposed by Hillinger (2002); see the Appendix for details.

4. Comparison

The traditional approach gives priority to the construction of sequences of values-at-constant-prices according to expression (5). Quantity changes between adjacent periods are then derived by expression (6). The modern approach gives priority to the computation of quantity index numbers for adjacent periods according to expression (10). Real values are then, according to expression (12), computed with help of chained index numbers. It looks like there are two distinct paradigms at stake here.

The core of Von der Lippe's critique (see footnote 1) appears to be that the properties of the sequence of real values (5) differ from those of (12), and that the properties of the Lowe quantity index (6) differ from those of the Fisher quantity index (10). Notably the fact that the real values computed according to (12) by chained index numbers are not additive, whereas the real values according to (5) do exhibit additivity, appears to be an important difference. Put otherwise, an important point of difference appears to be the fact that chained index numbers (can) exhibit behavior different from direct index numbers.

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[6] The practice in other countries is to use chained Paasche price index numbers and Laspeyres quantity index numbers respectively, as was recommended by Al *et al.* (1986); see also De Boer *et al.* (1997). *ESA 1995* considers this as acceptable. The use of chained Fisher index numbers was already mentioned by *SNA 1968*, par. 4.47.

It is relatively simple to show that this dispute cannot be solved. The first core question is, whether there exists a quantity index $Q(t,t')$ such that

$$Q(t,t') = Q(t,0)/Q(t',0). \tag{16}$$

Such a quantity index, that exhibits the property of circularity, should then necessarily be of the form

$$Q(t,t') = f(t)/f(t'). \tag{17}$$

The fundamental requirement that $Q(t,t') = 1$ if the quantity vectors of both periods are equal, leads then to the conclusion that *f(t)* must be a function of the quantities q^t only. Hence, prices p^t and $p^{t'}$ do not play any role in $Q(t,t')$.[7] One of course could accept this, be it that this implies that the price component corresponding to $Q(t,t')$, $(p^t \cdot q^t / p^{t'} \cdot q^{t'})/Q(t,t')$, does *not* pass the fundamental Identity Test; that is, if the price vectors of both periods are equal, then the last expression not necessarily delivers the outcome 1.

The second core question concerns the additivity, or, more generally, the consistency-in-aggregation, of price and quantity indexes. Suppose that our aggregate can be partitioned into *K* subaggregates and let (after permutation of commodities) the price and quantity vectors be partitioned as $p^t = (p_1^t,...,p_K^t)$ and $q^t = (q_1^t,...,q_K^t)$ respectively, where (p_k^t, q_k^t) is the subvector corresponding to the subaggregate *k* = 1,…,*K*. Let $P_k(t,t')$ be a price index with the same functional form as $P(t,t')$, but with its number of variables reduced to the number of commodities of subaggregate *k*. Similarly, let $Q_k(t,t')$ be a quantity index with the same functional form as $Q(t,t')$, but with its number of variables reduced to the number of commodities of subaggregate *k*. Now the real values computed according to (12) are called *additive* if

$$\sum_{k=1}^{K} \frac{p_k^t \cdot q_k^t}{P_k(t,0)} = \frac{p^t \cdot q^t}{P(t,0)}, \tag{18a}$$

that is, if real subaggregate values add up to the real aggregate value. In terms of quantity indexes additivity means that

$$\sum_{k=1}^{K} p_k^0 \cdot q_k^0 Q_k(t,0) = p^0 \cdot q^0 Q(t,0). \tag{18b}$$

The more general concept of consistency-in-aggregation for price and quantity indexes was defined by Balk (1995), (1996)[8]. A price index $P(t,t')$ is called consistent-in-aggregation if

[7] A more formal proof was given by Balk (1995).

[8] Pursiainen (2004) proposes a more general definition of consistency-in-aggregation. Restricted to indexes, however, his definition appears to reduce to the one presented here.

$$\sum_{k=1}^{K} \psi(P_k(t,t'), p_k^t \cdot q_k^t, p_k^{t'} \cdot q_k^{t'}) = \psi(P(t,t'), p^t \cdot q^t, p^{t'} \cdot q^{t'}), \tag{19a}$$

where $\psi(.)$ is a function that is continuous and strictly monotonous in its first variable. Likewise, a quantity index $Q(t,t')$ is called consistent-in-aggregation if

$$\sum_{k=1}^{K} \zeta(Q_k(t,t'), p_k^t \cdot q_k^t, p_k^{t'} \cdot q_k^{t'}) = \zeta(Q(t,t'), p^t \cdot q^t, p^{t'} \cdot q^{t'}), \tag{19b}$$

where $\zeta(.)$ is a function that is continuous and strictly monotonous in its first variable.

There are many, in fact infinitely many, functional forms for price and quantity indexes that satisfy (19a) or (19b). As an example, the reader is invited to consider the generalized mean price index $P(t,t') = [\sum_{n=1}^{N} (v_n^{t'} / V^{t'})(p_n^t / p_n^{t'})^{\rho}]^{1/\rho}$ where $\rho \neq 0$. Problems arise as soon as a number of very basic requirements are imposed on the price and quantity indexes. Specifically, it is assumed that

- the price and quantity indexes satisfy the Product Test (12);
- the price index satisfies the Equality Test; that is, if all the subaggregate price index numbers are equal, $P_k(t,t') = \lambda$ for all k = 1,…,K, then the aggregate price index number takes on the same magnitude, $P(t,t') = \lambda$;
- the quantity index satisfies the Equality Test; that is, likewise, if $Q_k(t,t') = \lambda$ for all k = 1,…,K, then $Q(t,t') = \lambda$;
- the price index $P(t,t')$ is linearly homogeneous in current period prices p^t;
- if the number of commodities in an aggregate reduces to 1, then the price index reduces to a price relative; that is, $P(t,t') = p^t / p^{t'}$ whenever N = 1.

Under these assumptions it can be shown that the only price indexes satisfying the consistency-in-aggregation requirement (19a) are those of Laspeyres and Paasche. It is now straightforward to check that any chained price index deviates from these two functional forms; *e.g.* for the chained Laspeyres price index one obtains

$$P_L^c(t,0) = \prod\nolimits_{\tau=1}^{t} \frac{p^{\tau} \cdot q^{\tau-1}}{p^{\tau-1} \cdot q^{\tau-1}} = \frac{p^* \cdot q^0}{p^0 \cdot q^0} \neq \frac{p^t \cdot q^0}{p^0 \cdot q^0}, \tag{20}$$

since

$$p^* = p^1 \prod\nolimits_{\tau=2}^{t} \frac{p^{\tau} \cdot q^{\tau-1}}{p^{\tau-1} \cdot q^{\tau-1}} \neq p^t. \tag{21}$$

Given this, mathematical, state-of-affairs it occurs to me as justified that priority is given to decomposing the value change between adjacent periods in a price and quantity index number. If one is to construct real values for a sequence of periods, then chained index numbers must be used for deflating or inflating. With the present day computation facilities and the basic data, however, it should be relatively simple, for analytical purposes, to compute alternative price and quantity index numbers, as well as alternative sequences of real values, among which values-at-constant-prices.

5. On the economic-theoretic interpretation of chained index numbers

It seems that the strategy of chaining is by and large motivated by practical reasons. The question considered in this, and the next, section is: what precisely does a chained index measure? This section approaches the question from the economic-theoretic point of view. For direct (bilateral) price and quantity indexes there is a well-established body of theory. Can this theory be used to provide an answer to our question?

5.1. Constant homothetic preference ordering

Suppose our price and quantity data (p^t, q^t) for $t = 0,1,\ldots,$T can be rationalized by a utility function; that is, there exists a continuous function $U(q)$, representing a preference ordering satisfying mild regularity conditions, such that

$$p^t \cdot q^t = C(p^t, U(q^t)), \qquad (22)$$

where $C(p,u) \equiv \min_q \{p \cdot q \mid U(q) \geq u\}$ is the cost function that is dual to $U(q)$. Duality theory tells us that $U(q)$ is homothetic if and only if the cost function can be decomposed as

$$C(p,u) = F(u)C(p,1) \equiv F(u)c(p), \qquad (23)$$

where *F(u)* is a function that is monotonously increasing in *u*, and $c(p)$ is called the unit cost function. Varian (1983), based on earlier work by Diewert (1973), showed that there exists a data rationalizing utility function which is homothetic if and only if a condition called the Homothetic Axiom of Revealed Preference (HARP) is satisfied. The specific form of this function is of no concern here.

As is well known, the Konüs cost of living index for period *t* relative to period *t'*, conditional on the utility level *u*, is defined by

$$P_K(t,t';u) \equiv \frac{C(p^t,u)}{C(p^{t'},u)} \qquad u \in Range(U). \qquad (24)$$

If the utility function is homothetic, then the Konüs cost of living index can be expressed as the ratio of values of the unit cost function, thus

$$P_K(t,t';u) = c(p^t)/c(p^{t'}) \equiv P_K(t,t') \qquad (25)$$

for any two periods *t, t'*. Using relations (25), (22), and the definition of the cost function, it is straightforward to derive the well-known Laspeyres and Paasche bounds:

$$P_K(t,t') = \frac{C(p^t,U(q^{t'}))}{C(p^{t'},U(q^{t'}))} \leq \frac{p^t \cdot q^{t'}}{p^{t'} \cdot q^{t'}} = P_L(t,t') \qquad (26)$$

$$P_K(t,t') = \frac{C(p^t, U(q^t))}{C(p^{t'}, U(q^t))} \geq \frac{p^t \cdot q^t}{p^{t'} \cdot q^t} = P_P(t,t'). \qquad (27)$$

Based on this double inequality, it is reasonable to consider the Fisher price index, $P_F(t,t') = [P_L(t,t')P_P(t,t')]^{1/2}$, as an approximation to the Konüs index $P_K(t,t')$. In fact, $P_F(t,t') = P_K(t,t')$ if and only if the unit cost function *c(p)* is quadratic (see Konüs and Byushgens 1926, Diewert 1976, Lau 1979).

There appear to be, however, many more bounds. Consider for instance an arbitrary third period $0 \leq s \leq T$. Then, by the same method, we find that also

$$P_K(t,t') = \frac{c(p^t)}{c(p^s)} \frac{c(p^s)}{c(p^{t'})} \leq \frac{p^t \cdot q^s}{p^s \cdot q^s} \frac{p^s \cdot q^{t'}}{p^{t'} \cdot q^{t'}} = P_L(t,s) P_L(s,t'), \qquad (28)$$

and

$$P_K(t,t') = \frac{c(p^t)}{c(p^s)} \frac{c(p^s)}{c(p^{t'})} \geq \frac{p^t \cdot q^t}{p^s \cdot q^t} \frac{p^s \cdot q^s}{p^{t'} \cdot q^s} = P_P(t,s) P_P(s,t'). \qquad (29)$$

The obvious generalization of this procedure is to consider all spanning trees connecting the periods 0,1,…,*T*. A spanning tree is a connected graph without cycles. Suppose that on such a tree the periods *t'* and *t* are connected via the periods *s(*2*),…,s(L*-1*)*, where $L \geq 3$, and call $t' = s(1)$ and $t = s(L)$; let *L*=2 represent the case where *t'* and *t* are adjacent (that is, the number of intermediate periods equals zero). Then

$$P_K(t,t') = \prod_{\ell=2}^{L} \frac{c(p^{s(\ell)})}{c(p^{s(\ell-1)})} \leq \prod_{\ell=2}^{L} P_L(s(\ell), s(\ell-1)). \qquad (30)$$

Taking the minimum of the right hand side of this expression over all spanning trees delivers the tightest upper bound for $P_K(t,t')$. Similarly, one obtains that

$$P_K(t,t') = \prod_{\ell=2}^{L} \frac{c(p^{s(\ell)})}{c(p^{s(\ell-1)})} \geq \prod_{\ell=2}^{L} P_P(s(\ell), s(\ell-1)), \qquad (31)$$

and taking the maximum of the right hand side of this expression over all spanning trees delivers the tightest lower bound for $P_K(t,t')$. Both tightest bounds can be computed by employing Warshall's algorithm. This algorithm at the same time checks whether HARP is satisfied and, if so, computes the tightest upper and lower bounds.

It is clear that, given that HARP is satisfied, the (direct) Laspeyres price index $P_L(t,t')$ as well as the chained Laspeyres price index $P_L^c(t,t')$ are elements of the set of upper bounds for the Konüs cost of living index $P_K(t,t')$; and that the (direct) Paasche price index $P_P(t,t')$ as well as the chained Paasche price index $P_P^c(t,t')$ are elements of the set of lower bounds. If $P_L^c(t,t') < P_L(t,t')$ then the chained Laspeyres price index is a tighter upper bound for the

Konüs index than the (direct) Laspeyres price index. Similarly, if $P_P^c(t,t') > P_P(t,t')$ then the chained Paasche price index is a tighter lower bound for the Konüs index than the (direct) Paasche price index.

We may conclude that, if both conditions are satisfied, then the chained Fisher price index $P_F^c(t,t') = [P_L^c(t,t')P_P^c(t,t')]^{1/2}$ is a better approximation to $P_K(t,t')$ than the (direct) Fisher price index.

5.2. Constant preference ordering

This nice result, however, only holds when HARP is satisfied. When HARP is not satisfied, it is still possible that there exists a data rationalizing utility function, that is, (22) holds; this function, however, is not necessarily homothetic. Varian (1982), based on earlier work by Afriat and Diewert (1973), showed this to be the case if and only if a condition called the Generalized Axiom of Revealed Preference (GARP) is satisfied. Under this weaker assumption the standard bounding result reads:

$$P_K(t,t';U(q^{t'})) \le P_L(t,t') \tag{32}$$
$$P_K(t,t';U(q^t)) \ge P_P(t,t') . \tag{33}$$

It can then be shown[9] that there exists a utility level u^* between $U(q^{t'})$ and $U(q^t)$ such that $P_K(t,t';u^*)$ lies between $P_L(t,t')$ and $P_P(t,t')$. Now $P_F(t,t')$ is a symmetric average of $P_L(t,t')$ and $P_P(t,t')$. Hence, if the interval between $P_L(t,t')$ and $P_P(t,t')$ is small, then one may expect that

$$P_F(t,t') \approx P_K(t,t';u^*) \text{ for some } u^* \text{ between } U(q^{t'}) \text{ and } U(q^t). \tag{34}$$

This is interesting, but not very useful if the periods *t* and *t'* are far apart and the difference between the Laspeyres and Paasche price index numbers is large. If this is the case, it is useful to consider the chained Fisher price index, which is built up from comparisons of adjacent periods. For such comparisons one may expect that the Laspeyres-Paasche spread is small enough to justify the use of (34). Hence,

$$P_F^c(t,t') = \prod_{\tau=t'+1}^{t} P_F(\tau,\tau-1) \approx \prod_{\tau=t'+1}^{t} P_K(\tau,\tau-1;u^{\tau*})$$
$$\text{for some } u^{\tau*} \text{ between } U(q^{\tau-1}) \text{ and } U(q^\tau) . \tag{35}$$

This result is still not very insightful. Equation (35) means that the chained Fisher price index approximates a chained Konüs index where the levels of utility vary over time. Somehow we should get rid of this variation. This can be accomplished by noticing that the Konüs index (24) is continuous in the utility level *u*. Choose $s = (t+t'+1)/2$ and assume that

$$P_K(\tau,\tau-1;u^{\tau*}) = P_K(\tau,\tau-1;U(q^s))\exp\{a(\tau-s)\} \text{ for some } a \ne 0 , \tag{36}$$

[9] The proof by Diewert (1981) goes back to Konüs.

which means that, conditional on prices p^{τ} and $p^{\tau-1}$, $P_K(\tau,\tau-1;u^{\tau*})$ is a loglinear function of the time variable associated with the reference utility level. By elementary analytical methods one can then show that

$$\prod_{\tau=t'+1}^{t} P_K(\tau,\tau-1;u^{\tau*}) = \prod_{\tau=t'+1}^{t} P_K(\tau,\tau-1;U(q^s)) = P_K(t,t';U(q^s)), \tag{37}$$

where the last equality follows from the transitivity of the Konüs index for fixed *u*. Thus, if (36) holds, then the chained Fisher price index $P_F^c(t,t')$ may be considered as approximating the Konüs cost of living index $P_K(t,t';U(q^s))$, where *s* is an intermediate time period. Notice that assumption (36) rules out any cycles.

5.3. Variable preference ordering

A still weaker, but not testable, assumption is that the preference ordering is changing over time, so that (22) must be replaced by

$$p^t \cdot q^t = C^t(p^t,U^t(q^t)) \tag{38}$$

where $U^t(q)$ represents the period *t* preference ordering and $C^t(p,u)$ its dual cost function. The Laspeyres and Paasche bounds still apply, but must be reformulated as

$$P_K^{t'}(t,t';U^{t'}(q^{t'})) \le P_L(t,t') \tag{39}$$
$$P_K^{t}(t,t';U^{t}(q^{t})) \ge P_P(t,t'). \tag{40}$$

A result such as (34), however, is now impossible, because the utility functions $U^{t'}(q)$ and $U^t(q)$ represent different preference orderings. Across periods it is meaningless to compare their numerical values.

There is a way out, however. A cost of living index including the preference change effect was defined by Balk (1989) as

$$P^{t,t'}(t,t';q) \equiv \frac{C^t(p^t,U^t(q))}{C^{t'}(p^{t'},U^{t'}(q))}. \tag{41}$$

This index conditions on the quantity vector *q* and compares the period *t* cost of the period *t* indifference class of *q* to the period *t'* cost of the period *t'* indifference class of *q*. It is a natural extension of the Konüs cost of living index: if the period *t* and *t'* preference orderings are identical, then $P^{t,t'}(t,t';q) = P_K(t,t';U(q))$. The index (41) can be decomposed into two parts, relating to the effect of price change and the effect of preference change. The effect of preference change is measured by $P^{t,t'}(t,t';q)$ by setting $p^t = p^{t'}$. This effect is not necessarily equal to 1, but, as argued by Balk (1989), has the right direction.

Balk (1989) also showed that the Laspeyres and Paasche bounds still apply, in the form:

$$P^{t,t'}(t,t';q^{t'}) \le P_L(t,t') \tag{42}$$
$$P^{t,t'}(t,t';q^{t}) \ge P_P(t,t'). \tag{43}$$

Then Diewert's (1981) proof can be used to show that there exists a quantity vector q^* between $q^{t'}$ and q^t such that $P^{t,t'}(t,t';q^*)$ lies between $P_L(t,t')$ and $P_P(t,t')$. If the interval between $P_L(t,t')$ and $P_P(t,t')$ is small, then one may expect that for the Fisher price index the following result holds:

$$P_F(t,t') \approx P^{t,t'}(t,t';q^*) \text{ for some } q^* \text{ between } q^{t'} \text{ and } q^t. \tag{44}$$

Assuming that for adjacent periods the Laspeyres-Paasche spread is indeed small, one obtains for the chained Fisher price index that

$$P_F^c(t,t') = \prod_{\tau=t'+1}^{t} P_F(\tau,\tau-1) \approx \prod_{\tau=t'+1}^{t} P^{\tau,\tau-1}(\tau,\tau-1;q^{\tau *})$$
$$\text{for some } q^{\tau *} \text{ between } q^{\tau-1} \text{ and } q^{\tau}. \tag{45}$$

The right hand side of this expression contains indexes that are conditional on quantity vectors that vary through time. Using the continuity of $P^{\tau,\tau-1}(\tau,\tau-1;q)$ in *q*, we assume that

$$P^{\tau,\tau-1}(\tau,\tau-1;q^{\tau *}) = P^{\tau,\tau-1}(\tau,\tau-1;q^{s})\exp\{b(\tau-s)\} \text{ for some } b \ne 0; \tag{46}$$

that is, conditional on prices p^τ and $p^{\tau-1}$, $P^{\tau,\tau-1}(\tau,\tau-1;q^{\tau *})$ is a loglinear function of the time variable associated with the reference quantity vector. Then, as in the previous subsection, one can show that

$$\prod_{\tau=t'+1}^{t} P^{\tau,\tau-1}(\tau,\tau-1;q^{\tau *}) = \prod_{\tau=t'+1}^{t} P^{\tau,\tau-1}(\tau,\tau-1;q^{s}) = P^{t,t'}(t,t';q^s), \tag{47}$$

where the last equality follows from the transitivity of (41) for fixed *q*. Thus, if (46) holds, then the chained Fisher price index $P_F^c(t,t')$ may be considered as approximating the cost of living index including the preference change effect $P^{t,t'}(t,t';q^s)$, where *s* is an intermediate time period. Notice that assumption (46) also rules out any cycles.

Recall that $P^{t,t'}(t,t';q^s)$ is not necessarily equal to 1 when $p^t = p^{t'}$. This feature is shared by a chained index such as $P_F^c(t,t')$. Put otherwise, the fact that a chained index violates the (bilateral) Identity Test reflects the fact that such an index encompasses the effect of preference change.

6. A look from Divisia index theory

For those who do not believe in well-behaved preference orderings and optimization, Divisia index theory might be used to shed light on the relation between direct and chained indices. This theory, however, requires a sort of mental leap: time periods must be considered as being of infinitesimal length and time itself as a continuous variable. Prices and quantities are supposed to be strictly positive, continuous and piecewise differentiable functions of time. Put otherwise, when time τ moves from period 0 to period *T*, prices and quantities $\langle p(\tau), q(\tau)\rangle$ describe a path through the 2*N*-dimensional, strictly positive, Euclidean orthant. It is also assumed that observations are available at periods 0, 1, 2, …, *T*; that is,

$$p(\tau) = p^{\tau} \text{ and } q(\tau) = q^{\tau} \text{ for } \tau = 0,1,...,T. \tag{48}$$

The starting point for Divisia index theory is the Product Test equation (12). It is straightforward to show, using elementary integral calculus, that this equation can be written as

$$p^{t}\cdot q^{t} / p^{0}\cdot q^{0} = p(t)\cdot q(t) / p(0)\cdot q(0) = P^{Div}(t,0)Q^{Div}(t,0) \quad (t = 1,\ldots,T) \tag{49}$$

where

$$\ln P^{Div}(t,0) \equiv \int_{\tau=0}^{t}\sum_{n=1}^{N} s_{n}(\tau)\frac{d\ln p_{n}(\tau)}{d\tau}d\tau, \tag{50}$$

$$\ln Q^{Div}(t,0) \equiv \int_{\tau=0}^{t}\sum_{n=1}^{N} s_{n}(\tau)\frac{d\ln q_{n}(\tau)}{d\tau}d\tau, \tag{51}$$

and

$$s_{n}(\tau) \equiv p_{n}(\tau)q_{n}(\tau) / p(\tau)\cdot q(\tau) \quad (n = 1,\ldots,N). \tag{52}$$

The problem is how to estimate these index numbers, given that one only has obtained observations on prices and quantities for a finite number of periods. Integral calculus provides us with the following two useful decompositions,

$$P^{Div}(t,0) \equiv \prod_{\tau=1}^{t} P^{Div}(\tau,\tau-1) \quad (t = 1,\ldots,T) \tag{53}$$

$$Q^{Div}(t,0) \equiv \prod_{\tau=1}^{t} Q^{Div}(\tau,\tau-1) \quad (t = 1,\ldots,T). \tag{54}$$

Now, as demonstrated by Balk (2000), for any pair of bilateral price and quantity indices $\langle P(t,t'), Q(t,t')\rangle$ there exists a (hypothetical) vector of functions $C \equiv \langle \hat{p}(\tau), \hat{q}(\tau)\rangle$, defined over the interval [*t'*,*t*] such that $\hat{p}(t') = p(t')$, $\hat{q}(t') = q(t')$, $\hat{p}(t) = p(t)$, and $\hat{q}(t) = q(t)$, and such that

$$P(t,t') = P_{C}^{Div}(t,t') \tag{55}$$

$$Q(t,t') = Q_{C}^{Div}(t,t'), \tag{56}$$

where the subscript *C* indicates that the integrals are computed using the functions defined by *C* rather than the true, but unknown functions occurring in (50) and (51). The closer one believes C to approximate these unknown functions, the better $\langle P(t,t'),Q(t,t')\rangle$ will approximate $\langle P^{Div}(t,t'),Q^{Div}(t,t')\rangle$. Balk's (2000) survey makes also clear that $\langle P_F(t,t'),Q_F(t,t')\rangle$ corresponds to a more reasonable price-quantity path than, say, $\langle P_P(t,t'),Q_L(t,t')\rangle$.

Given this theoretical knowledge, there are two distinct ways of approximating $\langle P^{Div}(t,0),Q^{Div}(t,0)\rangle$. The first is by calculating direct index numbers $\langle P_F(t,0),Q_F(t,0)\rangle$, which use only the period 0 and t data and hypothesize a path over the whole time interval. The second is, according to expressions (53) and (54), by calculating chained index numbers $\langle P_F^c(t,0),Q_F^c(t,0)\rangle$; these index numbers use also data of intermediate periods and hypothesize a segmented path that at the observation periods coincides with the true, but unknown, path. It may be clear that the second option should be preferred, since all available observations are being used and the hypothesized path stays closer to the true one.

7. Conclusion

By way of conclusion I return to the main problem, that of decomposing a value ratio in a price and a quantity component. Let $\langle P(t,t'),Q(t,t')\rangle$ be a pair of bilateral price and quantity indices that satisfy the Product Test. Then we have for any period $t = 2, \ldots, T$ the choice between the decompositions

$$V^t/V^0 = P(t,0)Q(t,0) \tag{57}$$

or

$$\begin{aligned} V^t/V^0 &= \prod_{\tau=1}^{t} V^\tau/V^{\tau-1} = \prod_{\tau=1}^{t} P(\tau,\tau-1)Q(\tau,\tau-1) \\ &= \prod_{\tau=1}^{t} P(\tau,\tau-1)\prod_{\tau=1}^{t} Q(\tau,\tau-1); \end{aligned} \tag{58}$$

that is, between using direct indices or chained indices. Notice, however, that expression (57) can easily be rewritten as

$$V^t/V^0 = \left[P(1,0)\prod_{\tau=2}^{t}\frac{P(\tau,0)}{P(\tau-1,0)}\right]\left[Q(1,0)\prod_{\tau=2}^{t}\frac{Q(\tau,0)}{Q(\tau-1,0)}\right], \tag{59}$$

the form of which is comparable to that of expression (58). From this point of view the question is not so much whether to decompose the value ratio between periods t and 0 by direct or chained indices, but whether adjacent periods should be compared by indices of the form $\langle P(\tau,0)/P(\tau-1,0),Q(\tau,0)/Q(\tau-1,0)\rangle$ or $\langle P(\tau,\tau-1),Q(\tau,\tau-1)\rangle$. Posed in this way, the answer seems obvious, because it is not at all clear why period 0 price and/or quantity data should play a role in the comparison of periods τ and $\tau-1$ ($\tau = 2, \ldots, t$).

As advanced in section 5.3, micro-economic theory suggests the use of Fisher indices for the comparison of adjacent periods, since in that case the chained price and quantity indices admit the interpretation of being approximations to cost of living and standard of living indices under changing preferences respectively. The main condition thereby is that the observed quantities do not exhibit cyclical behavior.

Appendix: A note on Hillinger's (2002) proposal

Hillinger (2002) proposes to replace chained Fisher price index numbers by chained Marshall-Edgeworth price index numbers, that is, to replace formula (13) by

$$P_{ME}^{c}(t,0) \equiv \prod\nolimits_{\tau=1}^{t} P_{ME}(\tau,\tau-1) \quad (t = 1,\ldots,T), \tag{A.1}$$

where the Marshall-Edgeworth price index is defined as

$$P_{ME}(t,t') = \frac{\frac{1}{2}(q^{t'} + q^{t})\cdot p^{t}}{\frac{1}{2}(q^{t'} + q^{t})\cdot p^{t'}} \quad (t = 1,\ldots,T). \tag{A.2}$$

This proposal has the disadvantage that the equality of deflation and inflation – see expression (12) – gets lost, since

$$\frac{p^{t}\cdot q^{t} / p^{0}\cdot q^{0}}{P_{ME}^{c}(t,0)} \neq Q_{ME}^{c}(t,0) \tag{A.3}$$

where $Q_{ME}^{c}(t,0)$ is a chained Marshall-Edgeworth quantity index, defined by (A.1) and (A.2) after interchanging prices and quantities. It appears that for two adjacent periods the quantity component

$$\frac{p^{t}\cdot q^{t} / p^{t-1}\cdot q^{t-1}}{P_{ME}(t,t-1)} = \frac{1+Q_{L}(t,t-1)}{1+1/Q_{P}(t,t-1)} \tag{A.4}$$

is dual to the "true factorial price index" and has the drawback of being not linearly homogeneous in q^{t} . Moreover, by mimicking the proof of Balk (1983) it is straightforward to show that the quantity index (A.4) is exact for a linear utility function.

The difference of two real values can be rewritten as

$$\frac{p^{t}\cdot q^{t}}{P_{ME}^{c}(t,0)} - \frac{p^{t-1}\cdot q^{t-1}}{P_{ME}^{c}(t-1,0)} =$$

$$\frac{1}{P_{ME}^{c}(t-1,0)}\left[\frac{1}{2}\left(\frac{p^{t}}{P_{ME}(t,t-1)} + p^{t-1}\right)\cdot\left(q^{t} - q^{t-1}\right) + \frac{1}{2}\left(q^{t} + q^{t-1}\right)\cdot\left(\frac{p^{t}}{P_{ME}(t,t-1)} - p^{t-1}\right)\right] =$$

$$\frac{1}{P_{ME}^{c}(t-1,0)}\left[\frac{1}{2}\left(\frac{p^{t}}{P_{ME}(t,t-1)} + p^{t-1}\right)\cdot\left(q^{t} - q^{t-1}\right)\right] =$$

$$\frac{1}{2}\left(\frac{p^t}{P_{ME}^c(t,0)}+\frac{p^{t-1}}{P_{ME}^c(t-1,0)}\right)\cdot\left(q^t-q^{t-1}\right), \tag{A.5}$$

where the next to last equality is based on definition (A.2). The difference of two real values can thus be written as a weighted average of individual quantity differences, $q_n^t - q_n^{t-1}$, which provides a nice interpretation.

The second component of Hillinger's (2002) proposal is to use the deflator (A.1) also for the computation of real values of subaggregates. The additivity problem is thereby not solved, but circumvented. Hillinger's argument is, however, not convincing. His proposal "appears to provide data users with very little information beyond what is already provided in the aggregates valued at current prices.", according to Ehemann *et al.* (2002). These authors also show that this proposal can lead to perverse outcomes.

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