



# **Economic Monotonicity of Price Index Formulas**

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## **Abstract:**

The present paper discusses, amends, and formalizes Kohli's (2005) notion of economic monotonicity. It relates this notion to axiomatic index theory. It is shown that the notion of strict monotonicity as defined in axiomatic index theory is somewhat different from Kohli's notion of economic monotonicity. However, the axiomatic approach provides useful suggestions for defining economic monotonicity in a precise manner. The present paper attempts to provide such a definition.

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## 1. Preliminary Remarks

The strict monotonicity test is probably one of the most widely accepted axioms developed in axiomatic index theory. In a paper included in this volume, Kohli (2005) implements the idea of monotonicity into an economic framework where quantities depend on prices. This is an important difference to axiomatic index theory. There it is assumed that quantities and prices are independent from each other. Kohli convincingly demonstrates that, embedded into such an economic framework, both the Paasche and Fisher index formulas violate the postulate of monotonicity. Since the Fisher index is often advocated as the most appropriate price index formula and the Paasche Index is widely used as GDP implicit price deflator, Kohli's findings challenge "general wisdom of index theory".

The present paper relates Kohli's approach to axiomatic index theory. It is shown that the notion of monotonicity as defined in axiomatic index theory is somewhat different from the notion of economic monotonicity. However, the axiomatic approach provides useful suggestions for defining economic monotonicity in a precise manner.

The paper proceeds as follows. In Section 2, the notion of monotonicity as defined in axiomatic index theory is presented. The notion of economic monotonicity is introduced in Section 3. In Section 4 it is examined whether the Laspeyres and Paasche index formulas satisfy economic monotonicity and Section 5 concludes.

## 2. Monotonicity in Axiomatic Index Theory

A primary concern of axiomatic index theory is the construction of appealing tests that can help to assess the reliability of price index formulas. Price index formulas that violate these tests must be viewed with suspicion and the results produced by them should be used with caution.

A price index formula  $P$  is a positive function that maps all of the prices and quantities in the base and comparison periods into a single positive number.

$$P : R_{++}^{4n} \rightarrow R_{++} , (p_0, q_0, p_1, q_1) \rightarrow P(p_0, q_0, p_1, q_1)$$

where

$$\mathbf{p}_t = (p_{1,t}, \dots, p_{N,t})^T \text{ and } \mathbf{q}_t = (q_{1,t}, \dots, q_{N,t})^T ,$$

$t$  is the time period,  $t=0$  (base period) and  $t=1$  (comparison period),

$p_{i,t} > 0$  is the unit price of commodity  $i$  in period  $t$ ,

$q_{i,t} > 0$  is the quantity of commodity  $i$  in period  $t$ ,

$i=1,2,\dots,N$ , where each integer  $i$  represents a particular commodity.

In the axiomatic approach, it is assumed that prices  $p_{i,t}$  and quantities  $q_{i,t}$  are independent from each other. Axiomatic index theory formulates tests that price index formulas should satisfy. A rather weak monotonicity test is the following:

**Definition 1** [Olt, 1996]: *If for all commodities  $p_{i,1} \geq p_{i,0}$  and for at least one  $i$  strict, then the weak monotonicity test postulates that*

$$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_1, \mathbf{q}_1) > P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_0, \mathbf{q}_1) .$$

Two decades before the weak monotonicity test the following test had been proposed:

**Definition 2** [Eichhorn and Voeller, 1976]: *Consider two different scenarios for the comparison period ( $t=1$  and  $t=1^*$ ) and the base period ( $t=0$  and  $t=0^*$ ). If for all commodities  $p_{i,1^*} \geq p_{i,1}$  and for at least one  $i$  strict, then the strict monotonicity test postulates that*

$$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_{1^*}, \mathbf{q}_1) > P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_1, \mathbf{q}_1) \tag{1}$$

*and if for all commodities  $p_{i,0^*} \geq p_{i,0}$  and for at least one  $i$  strict, then the strict monotonicity test postulates that*

$$P(\mathbf{p}_{0^*}, \mathbf{q}_0, \mathbf{p}_1, \mathbf{q}_1) < P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_1, \mathbf{q}_1) . \tag{2}$$

For the special  $\mathbf{p}_1 = \mathbf{p}_0$  inequality (1) simplifies to the weak monotonicity test.

### 3. Monotonicity in Economic Approaches to Index Theory

Kohli's (2005) concept of *economic monotonicity* relates to the strict monotonicity test. More specifically, it relates to the postulate expressed by inequality (1) and not to the postulate

expressed by inequality (2). Postulate (1) considers alternative scenarios for the comparison period. In contrast, postulate (2) considers alternative scenarios for the base period. Taking a more symmetric approach, that is, considering also postulate (2) as a necessary condition for monotonicity, appears as a useful amendment to Kohli's concept of economic monotonicity.

There is a second important difference between Kohli's notion of economic monotonicity and the strict monotonicity test (Definition 2). In Kohli's economic framework quantities are functions of prices:  $\mathbf{q}_0 = \mathbf{q}(\mathbf{p}_0, \mathbf{x}_0)$  and  $\mathbf{q}_1 = \mathbf{q}(\mathbf{p}_1, \mathbf{x}_1)$ , where the elements of the vectors  $\mathbf{x}_0$  and  $\mathbf{x}_1$  have exogenously given values. In the axiomatic approach, the quantity vectors  $\mathbf{q}_0$  and  $\mathbf{q}_1$  are treated as being *independent* from the price vectors  $\mathbf{p}_0$  and  $\mathbf{p}_1$  (and from  $\mathbf{x}_0$  and  $\mathbf{x}_1$ ).

A formal definition of the economic monotonicity axiom (including the symmetric treatment of base and comparison period scenarios) can be given. Suppose that

$$q_{i,t} = q(\mathbf{p}_t, \mathbf{x}_t) \text{ with either } \partial q_{i,t} / \partial p_{i,t} \geq 0 \text{ (for all } i = 1, 2, \dots, N; t = 0, 1) \quad (3a)$$

$$\text{or } \partial q_{i,t} / \partial p_{i,t} \leq 0 \text{ (for all } i = 1, 2, \dots, N; t = 0, 1) \quad (3b)$$

where  $\mathbf{x}_t$  represents a vector of variables with exogenously given values.

**Definition 3:** Consider two different scenarios for the comparison period ( $t=1$  and  $t=1^*$ ) and the base period ( $t=0$  and  $t=0^*$ ). If for all commodities  $p_{i,1^*} \geq p_{i,1}$  and for at least one  $i$  strict, then the *economic monotonicity test* postulates that

$$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_{1^*}, \mathbf{q}_{1^*}) > P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_1, \mathbf{q}_1) \quad (4)$$

and if for all commodities  $p_{i,0^*} \geq p_{i,0}$  and for at least one  $i$  strict, then the *economic monotonicity test* postulates that

$$P(\mathbf{p}_{0^*}, \mathbf{q}_{0^*}, \mathbf{p}_1, \mathbf{q}_1) < P(\mathbf{p}, \mathbf{q}_0, \mathbf{p}_1, \mathbf{q}_1) \quad (5)$$

The crucial difference between economic monotonicity (Definition 3) and strict monotonicity (Definition 2) is the following: In postulate (4)  $\mathbf{p}_{1^*}$  differs from  $\mathbf{p}_1$  and simultaneously  $\mathbf{q}_{1^*}$  is allowed to differ from  $\mathbf{q}_1$ , whereas in (1)  $\mathbf{q}_1$  is kept fixed. Analogously, in postulate (5)  $\mathbf{p}_{0^*}$  differs from  $\mathbf{p}_0$  and simultaneously  $\mathbf{q}_{0^*}$  is allowed to differ from  $\mathbf{q}_0$ , whereas in (2)  $\mathbf{q}_0$  is kept fixed.

#### 4. Laspeyres and Paasche Index

Kohli (2005) has demonstrated that the Paasche and Fisher index formulas violate economic monotonicity. Taking the approach stated in postulates (4) and (5), not only the Paasche and the Fisher index formulas but also the Laspeyres index formula violates economic monotonicity. In order to see why the Laspeyres Index violates economic monotonicity, it is useful to reformulate this index as the weighted arithmetic mean of price ratios, where the weights are “expenditure shares” of the base period:

$$\begin{aligned}
 P_{1,0}^L &= \frac{\sum_i p_{i,1} q_{i,0}}{\sum_i p_{i,0} q_{i,0}} \\
 &= \sum_i \frac{p_{i,0} q_{i,0}}{\sum_j p_{j,0} q_{j,0}} \frac{p_{i,1}}{p_{i,0}} \quad (6)
 \end{aligned}$$

This formula violates inequality (5) of economic monotonicity. Suppose, for example, that the price of only one good  $i$  differs between base period  $t=0$  and base period  $t=0^*$ . Inequality (5) postulates that for  $p_{i,0} < p_{i,0^*}$ , the Laspeyres Index must produce  $P_{1,0}^L > P_{1,0^*}^L$ . However, the Laspeyres Index does not necessarily produce  $P_{1,0}^L > P_{1,0^*}^L$ . If the case defined by relationship (3a) applies, that is, if in the applied economic framework all quantities are non-negatively related to prices, then the situation  $p_{i,0} < p_{i,0^*}$  implies that  $q_{i,0} \leq q_{i,0^*}$ . As a consequence, the weight  $p_{i,0} q_{i,0} / \sum_j p_{j,0} q_{j,0}$  is possibly much smaller than the weight  $p_{i,0^*} q_{i,0^*} / \sum_j p_{j,0^*} q_{j,0^*}$ . Therefore, in formula (6), the (larger) price increase ( $p_{i,1} / p_{i,0}$ ) receives a *smaller* weight than the (smaller) price increase ( $p_{i,1} / p_{i,0^*}$ ). In extreme cases, the difference in the weights overcompensates the difference in the respective price changes. This results in a less pronounced price index increasing effect of good  $i$  on the overall Laspeyres index:  $P_{1,0}^L < P_{1,0^*}^L$ . The latter, in turn, represents a violation of economic monotonicity.

The same line of reasoning can be applied to the Paasche Index. This index formula can be reformulated as the weighted harmonic mean of price ratios, where the weights are “expenditure shares” of the comparison period:

$$\begin{aligned}
P_{1,0}^L &= \frac{\sum_i P_{i,1} q_{i,1}}{\sum_i P_{i,0} q_{i,1}} \\
&= \left( \sum_i \frac{P_{i,1} q_{i,1}}{\sum_j P_{j,1} q_{j,1}} \left( \frac{P_{i,1}}{P_{i,0}} \right)^{-1} \right)^{-1} \tag{7}
\end{aligned}$$

This formula violates inequality (4) of economic monotonicity. Suppose, for example, that the price of only one good  $i$  differs between comparison period  $t = 1$  and base period  $t = 1^*$ . Inequality (4) postulates that for  $p_{i,1} < p_{i,1^*}$ , the Paasche Index must produce  $P_{1,0}^P < P_{1^*,0}^P$ . If quantity is non-negatively related to price, the situation  $p_{i,1} < p_{i,1^*} < p_{i,0}$  implies that  $q_{i,1} \leq q_{i,1^*}$ . As a consequence, the weight  $p_{i,1} q_{i,1} / \sum_j p_{j,1} q_{j,1}$  is possibly much smaller than the weight  $p_{i,1^*} q_{i,1^*} / \sum_j p_{j,1^*} q_{j,1^*}$ . Therefore, in formula (7), the (larger) price decline  $(p_{i,1} / p_{i,0})$  receives a smaller weight than the (smaller) price decline  $(p_{i,1^*} / p_{i,0})$ . In extreme cases, the difference in the weights overcompensates the difference in the respective price changes. This results in a less pronounced price index reducing effect of good  $i$  on the overall Paasche index:  $P_{1,0}^P > P_{1^*,0}^P$ .

In demonstrating that the Laspeyres and the Paasche index violate the economic monotonicity test, it was assumed that quantities are positively related to prices. What would happen, if quantities were negatively related to prices? This case is described by relationship (3b) and it is the standard case in the context of consumer theory? Would the price index formulas' violations of economic monotonicity vanish? They would not. The Laspeyres Index still violates postulate (5) and the Paasche Index postulate (4). The scope for violations, however, is much smaller than in the case of positively related quantities and prices. Violations require that price changes trigger extreme quantity changes. Only with these extreme quantity changes, the difference in weights can possibly overcompensate the difference in price changes.

## 5. Concluding Remarks

Axiomatic index theory has developed a large number of axioms. Among these axioms, the strict monotonicity test is certainly one of the most widely accepted axioms. Kohli (2005) has

introduced the concept of monotonicity into an economic framework. The present paper has shown that the notion of economic monotonicity can be formalized in a way that is very similar to the formalisation of the strict monotonicity test.

Kohli has demonstrated that the Paasche and Fisher index formulas violate economic monotonicity. The present paper has shown that the same deficiency applies to the Laspeyres index formula.

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